Elementary Probability

Maths 152 ©

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Community College of Philadelphia: SPRING 2004

Contents

Preface

These notes started during the Spring of 2002. The contents are mostly discrete probability, suitable for students who have mastered only elementary algebra.

Since a great number of the audience of this course comprises future school teachers, I have included a great deal of preliminary ancillary material.

Chapter 1

Preliminaries

1.1 Sets

1 Definition We define a *set* naively as a well-defined collection of objects. These objects are called the *elements* of the set.

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will have fixed meanings. The set of natural numbers is

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The set of integers is

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set of rational numbers (fractions of two integers, with non-zero denominator) is denoted by the symbol $\mathbb{Q}$. The set of real numbers is denoted by the symbol $\mathbb{R}$. The set of complex numbers is denoted by the symbol $\mathbb{C}$.

2 Definition We denote the set which has no elements, that is, the *empty* set by the symbol $\varnothing$.

3 Definition Let A be a set. If a belongs to the set A, then we write $a \in A$, read "a is an element of A" or "a is in A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A."

4 Example We have $-1 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$. That is, -1 is an integer, but $\frac{1}{2}$ is not.

5 Example Let $A = \{x \in \mathbb{Z} : x^2 = 4\}$. [1] Then clearly $A = \{-2, 2\}$.

6 Example Let $A = \{x \in \mathbb{Z} : x^2 = -4\}$. Then clearly $A = \varnothing$, since the square of an integer cannot be negative.

7 Definition If every element $a \in A$ is also an element of B, then we write $A \subseteq B$, which we read "A is a subset of B" or "A is contained in B." Two sets A, B are equal, written $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. If there is an $a \in A$ that does not belong to B then we write $A \nsubseteq B$, read "A is not a subset of B." If $A \subseteq B$ but $A \neq B$ then we write $A \subset B$, and we say that "A is a proper subset of B."

In particular for any set A we have $\varnothing \subseteq A$ and $A \subseteq A$.

8 Example Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

9 Example Let $A = \{-1, 0, 1\}$, $B = \{-1, 1\}$, and $C = \{1, 2\}$. Then $B \subset A$, $A \nsubseteq B$, $B \nsubseteq C$, $C \nsubseteq A$ and $C \nsubseteq B$.

10 Definition The *cardinality* of a set A, denoted by $\text{card}(A)$ is the number of elements that it has. If the set X has infinitely many elements, we write $\text{card}(X) = \infty$.

11 Example If $A = \{-1, 1\}$ then $\text{card}(A) = 2$. Also, $\text{card}(\mathbb{N}) = \infty$.

12 Definition The set of all subsets of a set A is the *power set* of A, denoted by $\mathscr{P}(A)$. In symbols

$$\mathscr{P}(A) = \{X : X \subseteq A\}.$$

[1]We read this as "A is the set of all x in $\mathbb{Z}$ such that x^2 equals 4.

13 Example If $A = \{0, 1, 2\}$ then

$$\mathscr{P}(A) = \{\varnothing, \{0\}, \{1\}, \{2\}, , \{0, 1\}, \{0, 2\}, \{1, 2\}, A\}.$$

We will prove in Theorem 186 that if $\mathrm{card}\,(A) = n < \infty$*, then* $\mathrm{card}\,(\mathscr{P}(A)) = 2^n$.

Homework

14 Problem Given the set $A = \{a, b\}$, find $\mathscr{P}(A)$ and $\mathrm{card}\,(\mathscr{P}(A))$.

Answer: $\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, A\}$ so $\mathrm{card}\,(\mathscr{P}(A)) = 4$.

15 Problem Let A be the set of all 3-element subsets of $\{1, 2, 3, 4\}$. List all the elements of A and find $\mathrm{card}\,(A)$.

Answer: $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and so $\mathrm{card}\,(A) = 4$.

1.2 Sample Spaces and Events

16 Definition A situation depending on chance will be called an *experiment*.

17 Example Some experiments in our probability context are

❶ rolling a die,

❷ flipping a coin,

❸ choosing a card from a deck,

❹ selecting a domino piece.

18 Definition A set $\Omega \neq \varnothing$ is called a *sample space* or *outcome space*. The elements of the sample space are called *outcomes*. A subset $A \subseteq \Omega$ is called an *event*. In particular, $\varnothing \subset \Omega$ is called the *null* or *impossible* event.

19 Example If the experiment is flipping a fair coin and recording whether heads H or tails T is obtained, then the sample space is $\Omega = \{H, T\}$.

20 Example If the experiment is rolling a fair die and observing how many dots are displayed, then the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. The event of observing an even number of dots is $E = \{2, 4, 6\}$ and the event of observing an odd number of dots is $O = \{1, 3, 5\}$. The event of observing a prime number score is $P = \{2, 3, 5\}$.

21 Example If the experiment consists of tossing two (distinguishable) dice (say one red, one blue), then the sample space consists of the 36 ordered pairs

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6), \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6), \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6), \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6), \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6), \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6). \end{array}$$

Here we record first the number on the red die and then the number on the blue die in the ordered pair (R, B). The event S of obtaining a sum of 7 is the set of ordered pairs

$$S = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

22 Example An experiment consists of the following two stages: (1) first a fair die is rolled and the number of dots recorded, (2) if the number of dots appearing is even, then a fair coin is tossed and its face recorded, and if the number of dots appearing is odd, then the die is tossed again, and the number of dots recorded. The sample space for this experiment is the set of 24 points

$$\begin{array}{ll} \{ & (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,H), (2,T), \\ & (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,H), (4,T), \\ & (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,H), (6,T) \quad \}. \end{array}$$

23 Example An experiment consists of drawing one card from a standard (52-card) deck and recording the card. The sample space is the set of 52 cards

$$\{ \quad A\clubsuit, 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit,$$
$$A\diamondsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit,$$
$$A\heartsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit,$$
$$A\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit \quad \}.$$

Homework

24 Problem An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.

Answer: $\{HH, HT, TH, TT\}$

25 Problem In the experiment of tossing two distinguishable dice in example 21, determine the event X of getting a product of 6, the event T of getting a sum smaller than 5, and the event U of getting a product which is a multiple of 7.

Answer: $X = \{(1, 6), (2, 3), (3, 2), (6, 1)\}$,
$T = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$, $U = \varnothing$

1.3 Combining Events

26 Definition The *union* of two events A and B, is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Observe that this "*or*" is inclusive, that is, it allows the possibility of x being in A, or B, or possibly both A and B. It should be clear that the following identities hold:

$$A \cup \varnothing = A, \tag{1.1}$$

$$A \cup A = A, \tag{1.2}$$

$$A \cup B = B \cup A, \tag{1.3}$$

$$(A \cup B) \cup C = A \cup (B \cup C). \tag{1.4}$$

We represent the union $A \cup B$ pictorially by the *Venn Diagram* in figure 1.1.

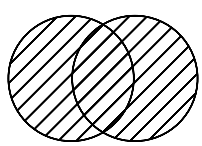


Figure 1.1: $A \cup B$

27 Example Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of example 20 and let $P = \{2, 3, 5\}$ be the event of getting a prime and $T = \{3, 6\}$ of getting a multiple of 3. Then the event of either getting a prime or a multiple of 6 in one toss of the die is $P \cup T = \{2, 3, 5, 6\}$.

28 Definition The *intersection* of two events A and B, is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The following identities hold:

$$A \cap \varnothing = \varnothing, \tag{1.5}$$

$$A \cap A = A, \tag{1.6}$$

$$A \cap B = B \cap A, \tag{1.7}$$

$$(A \cap B) \cap C = A \cap (B \cap C), \tag{1.8}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \tag{1.9}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{1.10}$$

We represent the intersection $A \cap B$ pictorially as in figure 1.2.

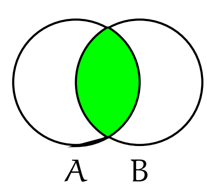


Figure 1.2: $A \cap B$

29 Example Consider the sample space of example 21, which are the 36 ordered pairs observed when rolling two distinguishable dice.

If S is the event of getting sum 7 and T is the event of getting product 12 then the event of simultaneously getting sum 7 and product 12 is

$$\begin{aligned} S \cap T &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \cap \{(2,6),(3,4),(4,3),(6,2)\} \\ &= \{(3,4),(4,3)\}. \end{aligned}$$

30 Definition Two events A and B are *disjoint* or *mutually exclusive* if $A \cap B = \varnothing$.

31 Definition The difference of events A *set-minus* B, is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

That is, $A \setminus B$ is all that which is contained in A but not in B.

We represent the difference $A \setminus B$ pictorially as in figure 1.3.

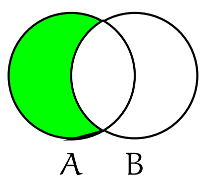


Figure 1.3: $A \setminus B$

By means of Venn Diagrams it should be clear that the following identities (called *De Morgan's Laws*) hold:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B), \tag{1.11}$$

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B). \tag{1.12}$$

We also have

$$X \setminus (X \setminus A) = X \cap A. \tag{1.13}$$

32 Example Let $A = \{1,2,3,4,5,6\}$, and $B = \{1,3,5,7,9\}$. Then

$$A \cup B = \{1,2,3,4,5,6,7,9\},$$

$$A \cap B = \{1,3,5\},$$

$$A \setminus B = \{2,4,6\},$$

and

$$B \setminus A = \{7,9\}.$$

33 Definition Let $A \subseteq X$. The *complement* of A with respect to X is $\complement A = X \setminus A$.

Observe that $\complement A$ is all that which is outside A. Usually we assume that A is a subset of some universal set U which is tacitly understood. The complement $\complement A$ represents the event that A does not occur. We represent $\complement A$ pictorially as in figure 1.4.

34 Example Let $U = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set of the decimal digits and let $A = \{0,2,4,6,8\} \subset U$ be the set of even digits. Then $\complement A = \{1,3,5,7,9\}$ is the set of odd digits.

Observe that

$$\complement A \cap A = \varnothing. \tag{1.14}$$

As a consequence of the De Morgan Laws, if A and B share the same universal set, we have

$$\complement(A \cup B) = \complement A \cap \complement B, \tag{1.15}$$

$$\complement(A \cap B) = \complement A \cup \complement B. \tag{1.16}$$

35 Example Let A, B, C be events. Then, as a function of A, B, C,

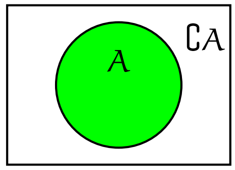


Figure 1.4: $\complement A$

❶ The event that only A happens is $A \cap \complement B \cap \complement C$.

❷ The event that only A and C happen, but not B is $A \cap \complement B \cap C$.

❸ The event that all three happen is $A \cap B \cap C$.

❹ The event that at least one of the three events occurs is $A \cup B \cup C$.

❺ The event that at least two of the three events occurs is

$$(A \cap B \cap \complement C) \cup (A \cap \complement B \cap C) \cup (\complement A \cap B \cap C) \cup (A \cap B \cap C).$$

❻ The event that at most one of the three events occurs is

$$(A \cap \complement B \cap \complement C) \cup (\complement A \cap \complement B \cap C) \cup (\complement A \cap \complement B \cap C) \cup (\complement A \cap \complement B \cap \complement C).$$

❼ The event that none of the events occurs is $\complement(A \cup B \cup C)$. Observe that by the De Morgan's Law this can also be written as $\complement A \cap \complement B \cap \complement C$.

❽ The event that exactly two of A, B, C occur is

$$(A \cap B \cap \complement C) \cup (A \cap \complement B \cap C) \cup (\complement A \cap B \cap C).$$

❾ The event that no more than two of A, B, C occur is $\complement(A \cap B \cap C)$.

36 Definition Let A be a non-empty set. A *partition* of A is a subdivision of A into non-empty, non-overlapping subsets whose union is A. Thus

37 Example Let $A = \{1,2,3,4\}$. Here are some partitions of A:

$$P_1 = \{\{1\},\{2\},\{3\},\{4\}\},$$

$$P_2 = \{\{1,2,3\},\{4\}\},$$

$$P_3 = \{\{1,4\},\{2,3\}\}.$$

On the other hand

$$N_1 = \{\{1,2\},\{2\},\{3\},\{4\}\},$$

is not a partition of A because $\{1,2\}$ and $\{2\}$ overlap, and

$$N_2 = \{\{1,2\},\{4\}\},$$

is not a partition of A because the element 3 is in none of the sets.

38 Example Figures 1.5, 1.6, 1.7 shew how to obtain $(X \setminus Z) \cup (Y \setminus Z)$. We first shade $X \setminus Z$ and $Y \setminus Z$, as in figures 1.5 and 1.6. We finally shade $(X \setminus Z) \cup (Y \setminus Z)$ as in figure 1.7.

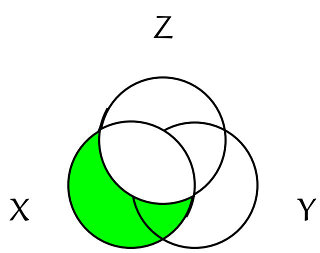


Figure 1.5: $X \setminus Z$

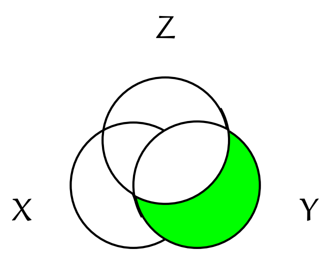


Figure 1.6: $Y \setminus Z$

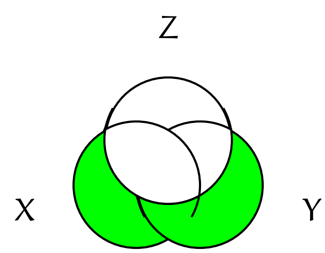


Figure 1.7: $(X\setminus Z)\cup(Y\setminus Z)$

Homework

39 Problem Let Ω be the set of married couples (men and women) in a certain town. Consider the events:

- A: "the man is older than 40",
- B: "the woman is younger than the man",
- C: "the woman is older than 40".

❶ Write in function of A, B, C the event: "the man is older than 40 but his wife is not."

❷ Describe in words the event $A \cap B \cap \complement C$.

❸ Describe in words the event $A \setminus (A \cap B)$.

❹ Describe in words the event $A \cap \complement B \cap C$.

❺ Describe in words the event $A \cup B$.

40 Problem Given sets X, Y, Z as follows.

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\},$$

$$Y = \{2, 4, 6, 8, 10, 12, 14, 16\},$$

$$Z = \{2, 3, 5, 7, 11, 13, 17\},$$

❶ Determine $X \setminus Z$.

❷ Determine $Y \setminus Z$.

❸ Determine $(X \setminus Z) \cap (Y \setminus Z)$.

❹ Determine $(X \setminus Z) \cup (Y \setminus Z)$.

❺ Determine $(X \setminus Z) \cup (Z \setminus X)$.

41 Problem Shade $\complement(A \cap B)$ and $\complement A \cup \complement B$ on figures 1.8 and 1.9 below. Similarly, shade $\complement(A \cup B)$ and $\complement A \cap \complement B$ on figures 1.10 and 1.11 below. What conclusion do the diagrams help you reach?

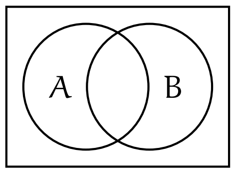


Figure 1.8: $\complement(A \cap B)$

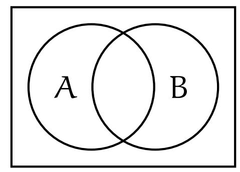


Figure 1.9: $\complement A \cup \complement B$

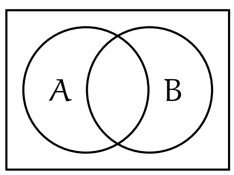


Figure 1.10: $\complement(A \cup B)$

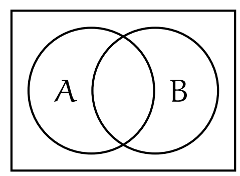
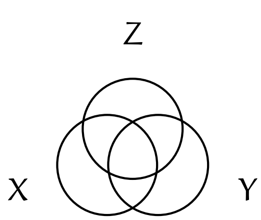
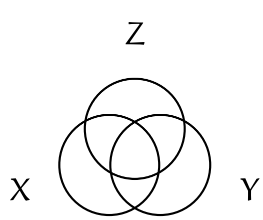
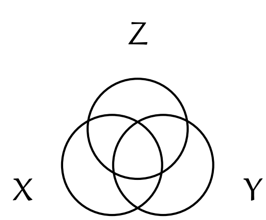


Figure 1.11: $\complement A \cap \complement B$

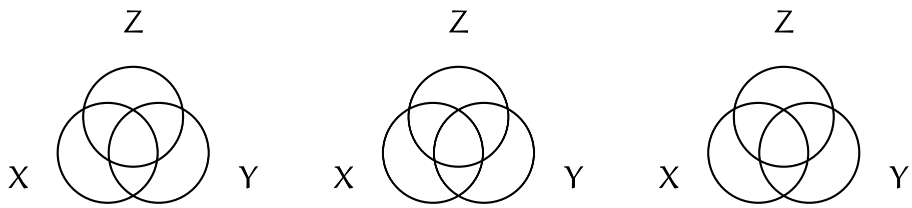
42 Problem Given X, Y, Z as shewn below. Use the diagrams in order to successively shade $X \setminus Z$, $Y \setminus Z$ and $(X \setminus Z) \cap (Y \setminus Z)$.







43 Problem Given X, Y, Z as shewn below. Use the diagrams in order to successively shade $X \setminus Z$, $Y \cap Z$ and $(X \setminus Z) \cup (Y \cap Z)$.



44 Problem Prove that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

We call $(A \cup B) \setminus (A \cap B)$ the *symmetric difference of* A *and* B and we write

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

See figure 1.12.

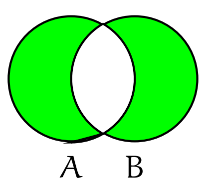


Figure 1.12: $A \triangle B$

1.4 The Integers

45 Definition Let a, b be integers with $a \neq 0$. Write $a|b$ (read "a divides b") if there exists an integer t such that $b = at$. We say that a is a *factor* of b and that b is a *multiple* of a.

For example $-5|10$ (-5 divides 10) because $10 = (-2)(-5)$. If c does not divide d we write $c \nmid d$.

46 Definition Let $a \in \mathbb{Z}$. The set of multiples of a is denoted by

$$a\mathbb{Z} = \{\ldots, -4a, -3a, -2a, -a, 0, a, 2a, 3a, 4a, \ldots, \}.$$

For example,

$$2\mathbb{Z} = \{\ldots - 8, -6, , -4, -2, 0, 2, 4, 6, 8, \ldots, \},$$

is the set of even integers and

$$3\mathbb{Z} = \{\ldots - 12, -9, , -6, -3, 0, 3, 6, 9, 12, \ldots, \},$$

is the set of multiples of 3.

47 Theorem Let a, b be integers, not both equal to 0. Then

$$a\mathbb{Z} \cap b\mathbb{Z} = \text{lcm}\,(a, b)\,\mathbb{Z}.$$

Proof If $x \in a\mathbb{Z} \cap b\mathbb{Z}$ then $x = as, x = bt$. Thus x is a common multiple of a and b. This means that $a\mathbb{Z} \cap b\mathbb{Z} \subseteq \text{lcm}\,(a, b)\,\mathbb{Z}$.

Conversely, there exist integers u, v such that $au = \text{lcm}\,(a, b)$ and $bv = \text{lcm}\,(a, b)$. Hence $\text{lcm}\,(a, b)\,\mathbb{Z} = au\mathbb{Z} \subseteq a\mathbb{Z}$ and $\text{lcm}\,(a, b)\,\mathbb{Z} = bv\mathbb{Z} \subseteq b\mathbb{Z}$. This means that $\text{lcm}\,(a, b)\,\mathbb{Z} \subseteq a\mathbb{Z} \cap b\mathbb{Z}$.

Since we have proved that $a\mathbb{Z} \cap b\mathbb{Z} \subseteq \text{lcm}\,(a, b)\,\mathbb{Z}$ and $\text{lcm}\,(a, b) = a\mathbb{Z} \cap b\mathbb{Z}$, we must conclude that $a\mathbb{Z} \cap b\mathbb{Z} = \text{lcm}\,(a, b)\,\mathbb{Z}$, as claimed.❑

48 Example

$$2\mathbb{Z}\cap 3\mathbb{Z} = \text{lcm}\,(2,3)\,\mathbb{Z} = 6\mathbb{Z},$$

$$12\mathbb{Z}\cap 15\mathbb{Z} = \text{lcm}\,(12,15)\,\mathbb{Z} = 60\mathbb{Z}.$$

49 Definition Let a, b be integers with $a \neq 0$. We define the set $a\mathbb{Z}+b$ as

$$a\mathbb{Z}+b = \{an+b : n \in \mathbb{Z}\}.$$

These are the integers that leave remainder b upon division by a.

Thus

$$2\mathbb{Z}+1 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

is the set of odd integers. Notice also that $2\mathbb{Z}+1 = 2\mathbb{Z}-1$.

$$3\mathbb{Z}+2 = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$$

is the set of integers leaving remainder 2 upon division by 3.

50 Definition Let x be a real number. The *floor* of x, denoted by $\lfloor x \rfloor$ is the greatest integer less than or equal to x. That is, $\lfloor x \rfloor$ is the unique integer satisfying the inequalities

$$x - 1 < \lfloor x \rfloor \leq x.$$

$\lfloor x \rfloor$ *is the integer* just to the left *of* x *if* x *is not an integer, and* x *if* x *is an integer.*

51 Example

$$\lfloor 0.5 \rfloor = 0,$$

$$\lfloor -0.5 \rfloor = -1,$$

$$\lfloor 2.2 \rfloor = 2,$$

$$\lfloor 2.9 \rfloor = 2,$$

$$\lfloor -2.2 \rfloor = -3,$$

$$\lfloor 2 \rfloor = 2.$$

52 Definition Let x be a real number. The *ceiling* of x, denoted by $\lceil x \rceil$ is the least integer greater than or equal to x. That is, $\lceil x \rceil$ is the unique integer satisfying the inequalities

$$x \leq \lceil x \rceil < x + 1.$$

$\lceil x \rceil$ is the integer just to the right *of x if x is not an integer, and x if x is an integer.*

53 Example

$$\lceil 0.5 \rceil = 1,$$
$$\lceil -0.5 \rceil = 0,$$
$$\lceil 2.2 \rceil = 3,$$
$$\lceil 2.9 \rceil = 3,$$
$$\lceil -2.2 \rceil = -2,$$
$$\lceil 2 \rceil = 2.$$

54 Example In the set $A = \{1, 2, \ldots, 500\}$ of 500 integers there are

$$\lfloor \frac{500}{2} \rfloor = 250 \quad \text{divisible by } 2\text{, namely } \{2, 4, 6, \ldots, 500\},$$
$$\lfloor \frac{500}{3} \rfloor = 166 \quad \text{divisible by } 3\text{, namely } \{3, 6, 9, \ldots, 498\},$$
$$\lfloor \frac{500}{5} \rfloor = 100 \quad \text{divisible by } 5\text{, namely } \{5, 10, 15, \ldots, 500\},$$
$$\lfloor \frac{500}{7} \rfloor = 71 \quad \text{divisible by } 7\text{, namely } \{7, 14, 21, \ldots, 497\},$$
$$\lfloor \frac{500}{11} \rfloor = 45 \quad \text{divisible by } 11\text{, namely } \{11, 22, 33, \ldots, 495\},$$
$$\lfloor \frac{500}{77} \rfloor = 6 \quad \text{divisible by } 7\text{, namely } \{77, 154, 231, \ldots, 462\},$$
$$\lfloor \frac{500}{251} \rfloor = 1 \quad \text{divisible by } 251\text{, namely } \{251\}.$$

55 Theorem (Division Algorithm) Let $a > 0$ be an integer. For every integer n there exist unique integers q and r such that

$$n = qa + r, \quad 0 \leq r < a.$$

Here a is the *divisor*, n the *dividend*, q the quotient, and r the *remainder.*

Proof n must lie between two consecutive multiples of a, that is, there exist q such that $qa \leq n < (q+1)a$. This gives

$$q \leq \frac{n}{a} < q + 1.$$

It follows that

$$q = \lfloor \frac{n}{a} \rfloor.$$

From this q is unique. We now let

$$r = n - qa = n - \lfloor \frac{n}{a} \rfloor a.$$

Clearly $0 \leq r < a$, and the uniqueness of r follows from that of q. ❑

There are exactly a possible remainders when an arbitrary integer is divided by a. Our version of the Division Algorithm says that these remainders may be either 0, or 1, or 2, ..., or $a - 1$.

56 Example For the divisor $a = 3$, we have

$$100 = 3(33) + 1,$$

$$101 = 3(33) + 2,$$

$$103 = 3(34) + 0,$$

$$-100 = 3(-34) + 2.$$

Notice that our version of the Division Algorithm requires that the remainder r satisfy $0 \leq r < 3$.

It is important to realise that given an integer $n > 0$, the Division Algorithm makes a partition of all the integers according to their remainder upon division by n. For example, every integer lies in one of the families $3k, 3k+1$ or $3k+2$ where $k \in \mathbb{Z}$. Observe that the family $3k+2, k \in \mathbb{Z}$, is the same as the family $3k-1, k \in \mathbb{Z}$. Thus

$$\mathbb{Z} = A \cup B \cup C$$

where

$$A = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$$

is the family of integers of the form $3k, k \in \mathbb{Z}$,

$$B = \{\ldots -8, -5, -2, 1, 4, 7, \ldots\}$$

is the family of integers of the form $3k+1, k \in \mathbb{Z}$ and

$$C = \{\ldots -7, -4, -1, 2, 5, 8, \ldots\}$$

is the family of integers of the form $3k-1, k \in \mathbb{Z}$.

Again, we can arrange all the integers in five columns as follows:

$$\begin{array}{rrrrr} \vdots & \vdots & \vdots & \vdots & \vdots \\ -10 & -9 & -8 & -7 & -6 \\ -5 & -4 & -3 & -2 & -1 \\ 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The arrangement above shews that any integer comes in one of 5 flavours: those leaving remainder 0 upon division by 5, those leaving remainder 1 upon division by 5, etc. We let

$$5\mathbb{Z} = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\},$$

$$
\begin{aligned}
5\mathbb{Z}+1 &= \{\ldots,-14,-9,-4,1,6,11,16,\ldots\},\\
5\mathbb{Z}+2 &= \{\ldots,-13,-8,-3,2,7,12,17,\ldots\},\\
5\mathbb{Z}+3 &= \{\ldots,-12,-7,-2,3,8,13,18,\ldots\},\\
5\mathbb{Z}+4 &= \{\ldots,-11,-6,-1,4,9,14,19,\ldots\}.
\end{aligned}
$$

57 Example Which number of $\{330,331,332,334,335,336,337,338,339\}$ lies in the sequence

$$-9,3,15,\ldots?$$

Solution: The numbers of the sequence have the form $12k+3, k = -1,0,1,2,\ldots,$, that is, they leave remainder 3 upon division by 12. Now, $339 = 12\cdot 28+3$, and so 339 is the only integer in the group that lies in the sequence.

58 Example Prove that if an integer leaves remainder 1 when divided by 12 then it also leaves remainder 1 when divided by 6. Prove also that the converse is not necessarily true.

Solution: An integer leaving remainder 1 when divided by 12 is of the form $12k+1 = 6(2k)+1 = 6t+1$, where $t = 2k$ is an integer. Thus it also leaves remainder 1 when divided by 6.

On the other hand, $7 = 6\cdot 1+1 = 12\cdot 0+7$ leaves remainder 1 when divided by 6 but remainder 7 when divided by 12. Another counterexample is $19 = 6\cdot 3+1 = 12\cdot 1+7$. In general, for the odd integer $2k+1$ ($k\in\mathbb{Z}$),

$$6(2k+1)+1 = 12k+7$$

leaves remainder 1 upon division by 6 but remainder 7 upon division by 12.

59 Example Prove that the square of an integer is either of the form $5k$, $5k+1$ or $5k+4$.

Solution: By the Division Algorithm, an integer has the form $5a$, $5a\pm 1$ or $5a\pm 2$. Now $(5a)^2 = 5(5a^2)$ is of the form $5k$. Also,

$$(5a\pm 1)^2 = 25a^2 \pm 10a+1 = 5(5a^2\pm 2a)+1$$

is of the form $5k+1$. Finally,

$$(5a\pm 2)^2 = 25a^2 \pm 20a + 4 = 5(5a^2 \pm 4a) + 4$$

is of the form $5k+4$.

60 Example (AHSME 1976) Let r be the remainder when $1059, 1417$ and 2312 are divided by $d > 1$. Find the value of $d - r$.

Solution: By the Division Algorithm,

$$1059 = q_1 d + r, 1417 = q_2 d + r, 2312 = q_3 d + r,$$

for some integers q_1, q_2, q_3. From this,

$$358 = 1417 - 1059 = d(q_2 - q_1), 1253 = 2312 - 1059 = d(q_3 - q_1)$$

and $895 = 2312 - 1417 = d(q_3 - q_2)$. Hence $d|358 = 2\cdot 179, d|1253 = 7\cdot 179$ and $7|895 = 5\cdot 179$. Since $d > 1$, we conclude that $d = 179$. Thus (for example) $1059 = 5\cdot 179 + 164$, which means that $r = 164$. We conclude that $d - r = 179 - 164 = 15$.

61 Example Shew that $n^2 + 23$ is divisible by 24 for infinitely many n.

Solution: $n^2 + 23 = n^2 - 1 + 24 = (n-1)(n+1) + 24$. If we take $n = 24k \pm 1, k = 0, 1, 2, \ldots$, all these values make the expression divisible by 24.

62 Example Prove that there are infinitely many integers n such that $4n^2 + 1$ is divisible by both 13 and 5.

Solution: Since $4n^2 + 1 = 4n^2 - 64 + 65 = 4(n-4)(n+4) + 65$, it suffices to take $n = \pm 4 + 65a,\ a \in \mathbb{Z}$.

Homework

63 Problem Determine the set $4\mathbb{Z} \cap 10\mathbb{Z}$.

Answer: $20\mathbb{Z}$

64 Problem Find $\lfloor \frac{100}{3} \rfloor$, $\lfloor \frac{3}{100} \rfloor$, $\lfloor -\frac{100}{3} \rfloor$, and $\lfloor -\frac{3}{100} \rfloor$.

Answer: $33, 0, -34, -1$.

65 Problem In the set of 600 integers $\{1, 2, \ldots, 600\}$ how many are divisible by 7? by 10? by 121?

Answer: 85; 60; 4

66 Problem In the set of 300 integers $\{2, 4, 6, \ldots, 600\}$ how many are divisible by 7? by 10? by 121?

Answer: 42; 30; 2

67 Problem Consider the arithmetic progression

$$-8, -3, 2, 7, \ldots.$$

Which of the 5 numbers $\{2000, 2001, 2002, 2003, 2004\}$, if any, belongs to it?

Answer: 2002

68 Problem Consider the arithmetic progression

$$-8, 12, 32, 52, \ldots.$$

Which of the 10 numbers

$$\{2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009\},$$

if any, belongs to it?

Answer: None.

69 Problem By the Division Algorithm, a number has one of the three forms $3k$, $3k+1$ or $3k+2$.

❶ Prove that if a number is divisible by 3 then its square is also divisible by 3.

❷ Prove that if a number leaves remainder 1 when divided by 3 then its square also leaves remainder 1 divided by 3.

❸ Prove that if a number leaves remainder 2 when divided by 3 then its square leaves remainder 1 divided by 3.

❹ What conditions must there be on X and Y so that $X^2 - Y^2$ be divisible by 3?

❺ What conditions must there be on X and Y so that $X^2 + Y^2$ be divisible by 3?

70 Problem Prove that the square of any integer is of the form $4k$ or $4k+1$. Use this to prove that numbers of the form $4k+3$ cannot be the sum of two squares. Deduce that 2003 is not the sum of two squares.

71 Problem Prove that n^3+5 is divisible by 4 for infinitely many values of n.

1.5 Divisibility Tests

In this section we study some divisibility tests. These will help us further classify the integers. We start with the simple

72 Theorem An integer n is divisible by 5 if and only if its last digit is a 0 or a 5.

Proof We derive the result for $n > 0$, for if $n < 0$ we simply apply the result to $-n > 0$. Now, let the decimal expansion of n be

$$n = a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0,$$

where $0 \leq a_i \leq 9,\ a_s \neq 0$. Then

$$n = 10(a_s 10^{s-1} + a_{s-1}10^{s-2} + \cdots + a_1) + a_0.$$

The first summand is divisible by 10 and it the divisibility of n by 5 thus depends on whether a_0 is divisible by 5, whence the result follows. ❑

73 Theorem Let k be a positive integer. An integer n is divisible by 2^k if and only if the number formed by the last k digits of n is divisible by 2^k.

Proof If $n = 0$ there is nothing to prove. If we prove the result for $n > 0$ then we can deduce the result for $n < 0$ by applying it to $-n = (-1)n > 0$. So assume that $n \in \mathbb{Z}$, $n > 0$ and let its decimal expansion be

$$n = a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0,$$

where $0 \leq a_i \leq 9,\ a_s \neq 0$. Now, each of $10^k = 2^k 5^k$, $10^{k+1} = 2^{k+1}5^{k+1}$, $\ldots$, $10^s = 2^s 5^s$, is divisible by 2^k, hence

$$\begin{aligned} n &= a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0 \\ &= 2^k(a_s 2^{s-k}5^s + a_{s-1}2^{s-k-1}5^{s-1} + \cdots + a_k 5^k) \\ &\qquad + a_{k-1}10^{k-1} + a_{k-2}10^{k-2} + \cdots + a_1 10 + a_0, \end{aligned}$$

so n is divisible by 2^k if and only if the number formed by the last k digits of n is divisible by 2^k. ❑

74 Example The number 987654888 is divisible by $2^3 = 8$ because the number formed by its last three digits, 888 is divisible by 8.

75 Example The number 191919191919193216 is divisible by $2^4 = 16$ because the number formed by its last four digits, 3216 is divisible by 16.

76 Example By what digits may one replace A so that the integer $231A2$ be divisible by 4?

Solution: The number $231A2$ is divisible by 4 if and only if $A2$ is divisible by 4. This happens when $A = 1$ $(A2 = 12)$, $A = 3$ $(A2 = 32)$, $A = 5$ $(A2 = 52)$, $A = 7$ $(A2 = 72)$, and $A = 9$ $(A2 = 92)$. Thus the five numbers

$$23112, 23132, 2315223172, 23192,$$

are all divisible by 4.

77 Example Determine digits a, b so that $235ab$ be divisible by 40.

Solution: $235ab$ will be divisible by 40 if and only if it is divisible by 8 and by 5. If $235ab$ is divisible by 8 then, *a fortiori*, it is even and since we also require it to be divisible by 5 we must have $b = 0$. Thus we need a digit a so that $5a0$ be divisible by 8. Since $0 \leq a \leq 9$, a quick trial an error gives that the desired integers are

$$23500, 23520, 23540, 23560, 23580.$$

78 Lemma If k is a positive integer, $9|(10^k - 1)$.

Proof This is immediate from the identity

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y^1 + x^{k-3}y^3 + \cdots + y^{k-1}),$$

upon putting $x = 10, y = 1$. ❑

79 Theorem (Casting-out 9's) An integer n is divisible by 9 if and only if the sum of it digits is divisible by 9.

Proof If $n = 0$ there is nothing to prove. If we prove the result for $n > 0$ then we can deduce the result for $n < 0$ by applying it to $-n = (-1)n > 0$. So assume that $n \in \mathbb{Z}$, $n > 0$ and let its decimal expansion be

$$n = a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0,$$

where $0 \leq a_i \leq 9$, $a_s \neq 0$. Now

$$\begin{aligned} n &= a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0 \\ &= a_s(10^s - 1) + a_{s-1}(10^{s-1} - 1) + \cdots + a_1(10 - 1) \\ &\qquad +a_s + \cdots + a_1 + a_0, \end{aligned}$$

from where the result follows. ❑

80 Example What values should the digit d take so that the number $32d5$ be divisible by 9?

Solution: The number $32d5$ is divisible by 9 if and only $3+2+d+5 = d+10$ is divisible by 9. Now,

$$0 \leq d \leq 9 \implies 10 \leq d + 10 \leq 19.$$

The only number in the range 10 to 19 divisible by 9 is 18, thus $d = 8$. One can easily verify that 3285 is divisible by 9.

Since $3|(10^k - 1)$ for positive integer k, we also obtained the following corollary.

81 Corollary An integer n is divisible by 3 if and only if its digital sum is divisible by 3.

82 Example Is there a digit d so that $125d$ be divisible by 45?

Solution: If $125d$ were divisible by 45, it must be divisible by 9 and by 5. If it were divisible by 5, then $d = 0$ or $d = 5$. If $d = 0$, the digital sum is $1 + 2 + 5 + 0 = 8$, which is not divisible by 9. Similarly, if $d = 5$, the digital sum is $1+2+5+5 = 13$, which is neither divisible by 9. So $125d$ is never divisible by 45.

83 Example (AHSME 1992) The two-digit integers from 19 to 92 are written consecutively in order to form the integer

$$192021222324 \cdots 89909192.$$

What is the largest power of 3 that divides this number?

Solution: By the casting-out-nines rule, this number is divisible by 9 if and only if

$$19 + 20 + 21 + \cdots + 92 = 37^2 \cdot 3$$

is. Therefore, the number is divisible by 3 but not by 9.

84 Definition If the positive integer n has decimal expansion

$$n = a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0,$$

the *alternating digital sum* of n is

$$a_s - a_{s-1} + a_{s-2} - a_{s-3} + \cdots + (-1)^{s-1} a_0$$

85 Example The alternating digital sum of 135456 is

$$1 - 3 + 5 - 4 + 5 - 6 = -2.$$

86 Lemma If t is even, then $11|(10^t - 1)$ and if t is odd, $11|(10^t + 1)$.

Proof Assume $t = 2a$, where a is a positive integer. Then

$$\begin{aligned} 10^{2a} - 1 &= (10^2 - 1)((10^2)^{a-1} + (10^2)^{a-2} + \cdots + 10^2 + 1) \\ &= 9 \cdot 11((10^2)^{a-1} + (10^2)^{a-2} + \cdots + 10^2 + 1), \end{aligned}$$

which is divisible by 11. Similarly if $t = 2a+1$, where $a \geq 0$ is an integer, then

$$\begin{aligned} 10^{2a+1}+1 &= (10+1)((10)^{2a}-(10)^{2a-1}+\cdots+10^2-10+1) \\ &= 11((10)^{2a}-(10)^{2a-1}+\cdots+10^2-10+1), \end{aligned}$$

which is again divisible by 11. ❑

87 Theorem An integer n is divisible by 11 if and only if its alternating digital sum is divisible by 11

Proof We may assume that $n > 0$. Let

$$n = a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0,$$

where $0 \leq a_i \leq 9,\ a_s \neq 0$. Assume first that s is even. Then

$$\begin{aligned} n &= a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0 \\ &= a_s(10^s - 1) + a_{s-1}(10^{s-1}+1) + a_{s-2}(10^{s-2}-1) + \cdots + a_1(10+1) \\ &\qquad + a_s - a_{s-1} + a_{s-2} \cdots - a_1 + a_0, \end{aligned}$$

and the result follows from this. Similarly, if s is odd,

$$\begin{aligned} n &= a_s 10^s + a_{s-1}10^{s-1} + \cdots + a_1 10 + a_0 \\ &= a_s(10^s + 1) + a_{s-1}(10^{s-1}-1) + a_{s-2}(10^{s-2}+1) + \cdots + a_1(10+1) \\ &\qquad - a_s + a_{s-1} - a_{s-2} \cdots - a_1 + a_0 \\ &= a_s(10^s + 1) + a_{s-1}(10^{s-1}-1) + a_{s-2}(10^{s-2}+1) + \cdots + a_1(10+1) \\ &\qquad -(a_s - a_{s-1} + a_{s-2} \cdots + a_1 - a_0), \end{aligned}$$

giving the result in this case. ❑

88 Example 912282219 has alternating digital sum $9-1+2-2+8-2+2-1+9=24$ and so 912282219 is not divisible by 11, whereas 8924310064539 has alternating digital sum $8-9+2-4+3-1+0-0+6-4+4-3+9=11$, and so 8924310064539 is divisible by 11.

Homework

89 Problem For which numbers $N \in \{1, 2, \ldots, 25\}$ will N^2+1 be divisible by 10?

Answer: $\{3, 7, 13, 17, 23\}$

90 Problem For which numbers $N \in \{1, 2, \ldots, 25\}$ will N^2-1 be divisible by 10?

Answer: $\{1, 9, 11, 19, 21\}$

91 Problem Determine a digit d, if at all possible, so that $2371d$ be divisible by 45.

Answer: $d = 5$

92 Problem Determine a digit d, if at all possible, so that $2371d$ be divisible by 44.

Answer: $d = 6$

93 Problem Determine a digit d, if at all possible, so that $23d3$ be divisible by 11.

Answer: There is no such digit.

94 Problem Determine a digit d, if at all possible, so that $653d7$ be divisible by 33.

95 Problem Find digits a, b, if at all possible, so that $1a2b4$ be divisible by 9.

96 Problem Find digits a, b, if at all possible, so that $1a2b4$ be divisible by 11.

97 Problem Why is it that no matter how you arrange the digits $0, 1, 2, \ldots, 9$ in order to form a 10-digit integer, the resulting integer is always divisible by 9?

98 Problem How must one arrange the digits $0, 1, 2, \ldots, 9$ in order to form a 10-digit integer divisible by 45?

Answer: The last digit must be 0 or 5, the other digits can be arranged at random.

99 Problem A *palindrome* is an integer whose decimal expansion is symmetric, and that does not end in 0. Thus

$$1, 2, 11, 101, 121, 9999, 123454321,$$

are all palindromes. Prove that a palindrome with an even number of digits is always divisible by 11.

100 Problem Shew that no matter how one distributes the digits $0, 1, 2, \ldots, 9$ in the blank spaces of

$$5\_383\_8\_2\_936\_5\_8\_203\_9\_3\_76,$$

the resulting number will always be divisible by 396.

1.6 Arithmetic Sums

101 Definition The sum $a_1 + a_2 + \cdots + a_n$ is denoted by

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

102 Example $\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4.$

103 Example $\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$

104 Example $\sum_{k=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 10.$

105 Example

$$\begin{aligned}\sum_{k=1}^{5} (2k-1) &= (1) + (2(2)-1) + (2(3)-1) + (2(4)-1) + (2(5)-1) \\ &= 1 + 3 + 5 + 7 + 9 \\ &= 25.\end{aligned}$$

106 Definition An *arithmetic progression* is one of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots.$$

Here a is the *first term* and d is the *common difference*. The n-th term is $a + (n-1)d$.

107 Example Find the 300-th term of the arithmetic progression

$$-9, 1, 11, 21, 31, \ldots.$$

Solution: Observe that the common difference is $1-(-9) = 11-1 = 21-11 = \ldots = 10$. The pattern is

$$-9,$$

$$1 = -9 + 1 \cdot 10,$$

$$21 = -9 + 2 \cdot 10,$$

$$31 = -9 + 3 \cdot 10,$$

etc. Hence the 300-th term is $-9 + 299(10) = 2981$.

108 Example Consider the progressions

$$P_1: \quad 4, 9, 14, \cdots, 499,$$

$$P_2: \quad 2, 5, 8, \cdots, 299.$$

How many elements do they have in common?

Solution: Observe that first progression has common difference 5 and the second has common difference 3. If there is a common element, there will be common elements in both separated by a distance of the least common multiple of 3 and 5, namely 15. Now observe that 14 is in both progressions. So we need

$$15k + 14 \leq 299 \implies k = 19.$$

Thus the $20 = 19 + 1$ elements

$$14 = 15 \cdot 0 + 14; 29 = 15 \cdot 1 + 14; 44 = 15 \cdot 2 + 14; \ldots; 299 = 15 \cdot 19 + 14$$

are in common.

109 Example Consider the progressions

$$P_1: \quad -9, 3, 15, \ldots, 1263,$$

$$P_2: \quad 7, 12, 17, \ldots, 502.$$

❶ Write a general formula for the elements of P_1.

❷ How many elements does P_1 have?

❸ Write a general formula for the elements of P_2.

❹ How many elements does P_2 have?

❺ Find the least positive integer that belongs to both progressions, if any.

❻ How many elements do they share?

Solution:

❶ The general term is $-9+12(n-1)$ for $n=1,2,\ldots$.

❷ We have

$$-9+12(n-1)=1263 \implies 12(n-1)=1272 \implies n=107.$$

❸ The general term is $7+5(n-1)$ for $n=1,2,\ldots$.

❹ We have

$$7+5(n-1)=502 \implies 5(n-1)=495 \implies n=100.$$

❺ Plainly this is 27.

❻ The overlapping elements have the form $27+60k$, $k=0,1,2,\ldots$. Thus we need

$$27+60k \leq 502 \implies k \leq \lfloor\frac{502-27}{60}\rfloor = 7.$$

Thus there are $7+1=8$ elements in common.

We are now interested in finding the sum of a finite arithmetic progression.

110 Theorem (Sum of a Finite Arithmetic Progression)

$$\begin{aligned}\sum_{k=1}^{n}(a+(k-1)d) &= (a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d)\\ &= \frac{n(2a+(n-1)d)}{2}.\end{aligned}$$

Proof Put

$$S=(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d).$$

Adding from the first to the last term is the same as adding from the last term to the first, so we have

$$S=(a+(n-1)d)+(a+(n-2)d)+(a+(n-3)d)+\cdots+(a).$$

Adding term by term, this gives

$$2S=(2a+(n-1)d)+(2a+(n-1)d)+(2a+(n-1)d)+\cdots+(2a+(n-1)d),$$

or

$$2S=n(2a+(n-1)d),$$

from where the theorem follows. ❑

111 Example Consider the following progression.

$$16, 20, 24, \ldots.$$

You may assume that this pattern is preserved.

❶ Find the common difference.

❷ Find a formula for the n-th term.

❸ Find the 100-th term of the progression.

❹ Find the sum of the first 100 terms of the progression.

Solution:

❶ The common difference is $+4$.

❷ The n-th term is $16+4(n-1), \quad n=1,2,3,\ldots$.

❸ The 100-th term is $16+4(99)=412$

❹ If
$$S=16+20+\cdots+412,$$
then
$$2S=(16+412)+(20+408)+\cdots+(412+16)=(428)(100),$$
whence $S=21400$.

One important arithmetic sum is
$$A_n=\sum_{k=1}^{n} k=1+2+\cdots+n.$$

By putting $a=1, d=1$ in Theorem 110, we obtain

$$\sum_{k=1}^{n} k=1+2+\cdots+n=\frac{n(n+1)}{2}.$$

112 Example

$$1+2+3+\cdots+100=\frac{100(101)}{2}=5050.$$

113 Example Find the sum of all the integers from 1 to 1000 inclusive, which are not multiples of 3 or 5.

Solution: We compute the sum of all integers from 1 to 1000 and weed out the sum of the multiples of 3 and the sum of the multiples of 5, but we put back the multiples of 15, which we have counted twice. Put
$$A_n=1+2+3+\cdots+n,$$

$$B = 3+6+9+\cdots+999 = 3(1+2+\cdots+333) = 3A_{333},$$

$$C = 5+10+15+\cdots+1000 = 5(1+2+\cdots+200) = 5A_{200},$$

$$D = 15+30+45+\cdots+990 = 15(1+2+\cdots+66) = 15A_{66}.$$

The desired sum is

$$\begin{aligned} A_{1000} - B - C + D &= A_{1000} - 3A_{333} - 5A_{200} + 15A_{66} \\ &= 500500 - 3\cdot 55611 - 5\cdot 20100 + 15\cdot 2211 \\ &= 266332. \end{aligned}$$

114 Example Each element of the set $\{10, 11, 12, \ldots, 19, 20\}$ is multiplied by each element of the set $\{21, 22, 23, \ldots, 29, 30\}$. If all these products are added, what is the resulting sum?

Solution: This is asking for the product $(10+11+\cdots+20)(21+22+\cdots+30)$ after all the terms are multiplied. But

$$10+11+\cdots+20 = \frac{(20+10)(11)}{2} = 165$$

and

$$21+22+\cdots+30 = \frac{(30+21)(10)}{2} = 255.$$

The required total is $(165)(255) = 42075$.

115 Example Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

Solution: We want the sum of the integers of the form $6r+2, r = 0, 1, \ldots, 16$. But this is

$$\sum_{r=0}^{16}(6r+2) = 6\sum_{r=0}^{16} r + \sum_{r=0}^{16} 2 = 6\frac{16(17)}{2} + 2(17) = 850.$$

Homework

116 Problem Find the sum $\sum_{k=1}^{5}(k^2+k+1)$.

117 Problem Find the sum $\sum_{k=1}^{3}\frac{k^2-1}{k^2+1}$.

118 Problem How many terms are shared by the progressions

$$P_1: \quad 5, 9, 13, \ldots, 405,$$

$$P_2: \quad 4, 9, 14, \ldots, 504?$$

119 Problem How many terms are shared by the progressions

$$P_1: \quad 5, 9, 13, \ldots, 405,$$

$$P_2: \quad 10, 19, 28, \ldots, 910?$$

120 Problem Consider the following progression.

$$98, 90, 82, \ldots$$

You may assume that this pattern is preserved.

❶ Find the common difference.

❷ Find the fourth term of the progression.

❸ Find the 51-st term of the progression.

❹ Find the sum of the first 51 terms of the progression.

121 Problem Consider the following progression.

$$a, a-b, a-2b, \ldots.$$

You may assume that this pattern is preserved.

❶ Find the common difference.

❷ Find the third term of the progression.

❸ Find the 101-st term of the progression.

❹ Find the sum of the first 101 terms of the progression.

Answer: $-b$; $a-2b$; $a-100b$; $101-5050b$

122 Problem Find a formula for the n-th term of the progression

$$a-2d,\ a-d, a, a+d, \ldots.$$

Then find the sum of the first 100 terms.

Answer: The n-th term is $a-2d+d(n-1)=a+d(n-3)$. The sum of the first 100 terms is $50(2a+95d)$.

123 Problem The consecutive odd integers are grouped as follows:

$$\{1\},$$
$$\{3,5\},$$
$$\{7,9,11\},$$
$$\{13,15,17,19\},$$
$$\vdots$$

Shew that the sum of the n-th group is n^3.

1.7 Geometric Sums

124 Definition A *geometric progression* is one of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots,$$

with $a \neq 0, r \neq 0$. Here a is the *first term* and r is the *common ratio*.

125 Example Find the 30-th term of the geometric progression

$$-\frac{3}{1024}, \frac{3}{512}, -\frac{3}{256}, \ldots.$$

Solution: The common ratio is

$$\frac{3}{512} \div \left(-\frac{3}{1024}\right) = -2.$$

Hence, the 30-th term is

$$\left(-\frac{3}{1024}\right)(-2)^{29} = \left(\frac{3}{2^{10}}\right)2^{29} = 3 \cdot 2^{19} = 1572864.$$

Let us sum now the geometric series

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Plainly, if $r = 1$ then $S = na$, so we may assume that $r \neq 1$. We have

$$rS = ar + ar^2 + \cdots + ar^n.$$

Hence

$$S - rS = a + ar + ar^2 + \cdots + ar^{n-1} - ar - ar^2 - \cdots - ar^n = a - ar^n.$$

From this we deduce that

$$S = \frac{a - ar^n}{1 - r},$$

that is,

$$a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r},$$

which yields

126 Theorem (Sum of a Finite Geometric Progression) Let $r \neq 1$. Then

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

127 Corollary (Sum of an Infinite Geometric Progression) Let $|r| < 1$. Then

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + \cdots + ar^{n-1} + \cdots = \frac{a}{1 - r}.$$

Proof If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$. The result now follows from Theorem 126. ❑

128 Example Find the following geometric sum:

$$1 + 2 + 4 + \cdots + 1024.$$

Solution: Let

$$S = 1 + 2 + 4 + \cdots + 1024.$$

Then

$$2S = 2 + 4 + 8 + \cdots + 1024 + 2048.$$

Hence

$$S = 2S - S = (2+4+8\cdots+2048) - (1+2+4+\cdots+1024) = 2048 - 1 = 2047.$$

129 Example Find the geometric sum

$$x = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}}.$$

Solution: We have

$$\frac{1}{3}x = \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}} + \frac{1}{3^{100}}.$$

Then

$$\begin{aligned}\frac{2}{3}x &= x - \frac{1}{3}x \\ &= \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}}\right) \\ &\quad - \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}} + \frac{1}{3^{100}}\right) \\ &= \frac{1}{3} - \frac{1}{3^{100}}.\end{aligned}$$

From which we gather

$$x = \frac{1}{2} - \frac{1}{2 \cdot 3^{99}}.$$

130 Example Find the sum

$$S_n = 1 + 1/2 + 1/4 + \cdots + 1/2^n.$$

Interpret your result as $n \to \infty$.

Solution: We have

$$S_n - \frac{1}{2}S_n = (1 + 1/2 + 1/4 + \cdots + 1/2^n) - (1/2 + 1/4 + \cdots + 1/2^n + 1/2^{n+1}) = 1 - 1/2^n.$$

Whence

$$S_n = 2 - 1/2^n.$$

So as n varies, we have:

$$\begin{aligned}
S_1 &= 2 - 1/2^0 &=& \quad 1 \\
S_2 &= 2 - 1/2 &=& \quad 1.5 \\
S_3 &= 2 - 1/2^2 &=& \quad 1.875 \\
S_4 &= 2 - 1/2^3 &=& \quad 1.875 \\
S_5 &= 2 - 1/2^4 &=& \quad 1.9375 \\
S_6 &= 2 - 1/2^5 &=& \quad 1.96875 \\
S_{10} &= 2 - 1/2^9 &=& \quad 1.998046875
\end{aligned}$$

Thus the farther we go in the series, the closer we get to 2.

131 Example Find the infinite geometric sum

$$\frac{10}{3} - \frac{20}{9} + \frac{40}{27} - \frac{80}{81} + \cdots .$$

Solution: The first term is $a = \frac{10}{3}$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| < 1$ we find in view of Theorem 127 that the sum is

$$\frac{a}{1-r} = \frac{\frac{10}{3}}{1 - \left(-\frac{2}{3}\right)} = 2.$$

132 Example A fly starts at the origin and goes 1 unit up, $1/2$ unit right, $1/4$ unit down, $1/8$ unit left, $1/16$ unit up, etc., *ad infinitum.* In what co-ördinates does it end up?

Solution: Its x co-ördinate is

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots = \frac{\frac{1}{2}}{1 - \frac{-1}{4}} = \frac{2}{5}.$$

Its y co-ördinate is

$$1-\frac{1}{4}+\frac{1}{16}-\cdots=\frac{1}{1-\frac{-1}{4}}=\frac{4}{5}.$$

Therefore, the fly ends up in

$$\left(\frac{2}{5},\frac{4}{5}\right).$$

The following example presents an *arithmetic-geometric* sum.

133 Example Sum

$$a=1+2\cdot 4+3\cdot 4^2+\cdots+10\cdot 4^9.$$

Solution: We have

$$4a=4+2\cdot 4^2+3\cdot 4^3+\cdots+9\cdot 4^9+10\cdot 4^{10}.$$

Now, $4a-a$ yields

$$3a=-1-4-4^2-4^3-\cdots-4^9+10\cdot 4^{10}.$$

Adding this last geometric series,

$$a=\frac{10\cdot 4^{10}}{3}-\frac{4^{10}-1}{9}.$$

Homework

134 Problem Find the sum

$$\frac{1}{2}+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^{10}\cdots.$$

135 Problem Find the sum

$$\frac{2}{3}+\frac{2}{3}\left(\frac{1}{3}\right)^2+\frac{2}{3}\left(\frac{1}{3}\right)^4+\frac{2}{3}\left(\frac{1}{3}\right)^6\cdots.$$

136 Problem Consider the following progression.

$$\frac{1}{625}, \frac{1}{125}, \frac{1}{25}, \ldots,$$

You may assume that this pattern is preserved.

❶ Find the common ratio.

❷ Find the fourth term of the progression.

❸ Find the 10-th term of the progression.

❹ Find the sum of the first 10 terms of the progression.

❺ Is it possible to find the infinite sum

$$\frac{1}{625} + \frac{1}{125} + \frac{1}{25} + \cdots ?$$

If it is, find it. If it is not, explain why.

Answer: 5; $\frac{1}{5}$; 3125 ; $\frac{2441406}{625}$; No, since the common ratio $5 > 1$.

137 Problem Let

$$n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 6, n_5 = 8, n_6 = 9, n_7 = 12, \ldots$$

be the sequence of positive integers whose prime factorisations consists of only 2's and 3's. Find

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \cdots .$$

Answer: 2

Chapter 2

Counting

2.1 Inclusion-Exclusion

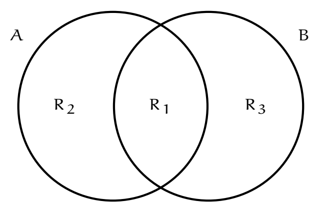


Figure 2.1: Two-set Inclusion-Exclusion

The Principle of Inclusion-Exclusion is attributed to both Sylvester and to Poincaré.

138 Theorem (Two set Inclusion-Exclusion)

$$\text{card}(A \cup B) = \text{card}(A) + \text{card}(B) - \text{card}(A \cap B)$$

Proof In the Venn diagram 2.1, we mark by R_1 the number of elements which are simultaneously in both sets (i.e., in $A \cap B$), by R_2 the number of elements which are in A but not in B (i.e., in $A \setminus B$), and by R_3 the number of elements which are B but not in A (i.e., in $B \setminus A$). We have $R_1 + R_2 + R_3 = \text{card}\,(A \cup B)$, which proves the theorem. ❑

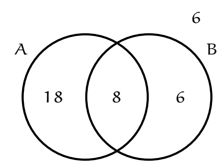


Figure 2.2: Example 139.

139 Example Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?

Solution: Let A denote the set of smokers and B the set of chewers. Then

$$\text{card}\,(A \cup B) = \text{card}\,(A) + \text{card}\,(B) - \text{card}\,(A \cap B) = 28 + 16 - 10 = 34,$$

meaning that there are 34 people that either smoke or chew (or possibly both). Therefore the number of people that neither smoke nor chew is $40 - 34 = 6$.
Aliter: We fill up the Venn diagram in figure 2.2 as follows. Since $|A \cap B| = 8$, we put an 10 in the intersection. Then we put a $28 - 10 = 18$ in the part that A does not overlap B and a $16 - 10 = 6$ in the part of B that does not overlap A. We have accounted for $10 + 18 + 6 = 34$ people that are in at least one of the set. The remaining $40 - 34 = 6$ are outside the sets.

140 Example Consider the set

$$A = \{2, 4, 6, \ldots, 114\}.$$

❶ How many elements are there in A?

❷ How many are divisible by 3?

❸ How many are divisible by 5?

❹ How many are divisible by 15?

❺ How many are divisible by either $3, 5$ or both?

❻ How many are neither divisible by 3 nor 5?

❼ How many are divisible by exactly one of 3 or 5?

Solution: Let $A_3 \subset A$ be the set of those integers divisible by 3 and $A_5 \subset A$ be the set of those integers divisible by 5.

❶ Notice that the elements are $2 = 2(1), 4 = 2(2), \ldots, 114 = 2(57)$. Thus card $(A) = 57$.

❷ There are $\lfloor \frac{57}{3} \rfloor = 19$ integers in A divisible by 3. They are

$$\{6, 12, 18, \ldots, 114\}.$$

Notice that $114 = 6(19)$. Thus card $(A_3) = 19$.

❸ There are $\lfloor \frac{57}{5} \rfloor = 11$ integers in A divisible by 5. They are

$$\{10, 20, 30, \ldots, 110\}.$$

Notice that $110 = 10(11)$. Thus card $(A_5) = 11$

❹ There are $\lfloor \frac{57}{15} \rfloor = 3$ integers in A divisible by 15. They are $\{30, 60, 90\}$. Notice that $90 = 30(3)$. Thus card $(A_{15}) = 3$, and observe that by Theorem 47 we have card $(A_{15}) =$ card $(A_3 \cap A_5)$.

❺ We want card $(A_3 \cup A_5) = 19 + 11 = 30$.

❻ We want

$$\text{card}\,(A \setminus (A_3 \cup A_5)) = \text{card}\,(A) - \text{card}\,(A_3 \cup A_5) = 57 - 30 = 27.$$

❼ We want

$$\begin{aligned}\text{card}\left((A_3 \cup A_5) \setminus (A_3 \cap A_5)\right) &= \text{card}\left((A_3 \cup A_5)\right) - \text{card}\left(A_3 \cap A_5\right) \\ &= 30 - 3 \\ &= 27.\end{aligned}$$

141 Example How many integers between 1 and 1000 inclusive, do not share a common factor with 1000, that is, are relatively prime to 1000?

Solution: Observe that $1000 = 2^3 5^3$, and thus from the 1000 integers we must weed out those that have a factor of 2 or of 5 in their prime factorisation. If A_2 denotes the set of those integers divisible by 2 in the interval $[1; 1000]$ then clearly $\text{card}\left(A_2\right) = \lfloor \frac{1000}{2} \rfloor = 500$. Similarly, if A_5 denotes the set of those integers divisible by 5 then $\text{card}\left(A_5\right) = \lfloor \frac{1000}{5} \rfloor = 200$. Also $\text{card}\left(A_2 \cap A_5\right) = \lfloor \frac{1000}{10} \rfloor = 100$. This means that there are $\text{card}\left(A_2 \cup A_5\right) = 500 + 200 - 100 = 600$ integers in the interval $[1; 1000]$ sharing at least a factor with 1000, thus there are $1000 - 600 = 400$ integers in $[1; 1000]$ that do not share a factor prime factor with 1000.

142 Theorem (Three set Inclusion-Exclusion)

$$\begin{aligned}\text{card}\left(A \cup B \cup C\right) &= \text{card}\left(A\right) + \text{card}\left(B\right) + \text{card}\left(C\right) \\ &\quad - \text{card}\left(A \cap B\right) - \text{card}\left(B \cap C\right) - \text{card}\left(C \cap A\right) \\ &\quad + \text{card}\left(A \cap B \cap C\right)\end{aligned}$$

Proof Using the associativity and distributivity of unions of sets, we see that

$$\begin{aligned}
\text{card}(A \cup B \cup C) &= \text{card}(A \cup (B \cup C)) \\
&= \text{card}(A) + \text{card}(B \cup C) - \text{card}(A \cap (B \cup C)) \\
&= \text{card}(A) + \text{card}(B \cup C) - \text{card}((A \cap B) \cup (A \cap C)) \\
&= \text{card}(A) + \text{card}(B) + \text{card}(C) - \text{card}(B \cap C) \\
&\quad -\text{card}(A \cap B) - \text{card}(A \cap C) \\
&\quad +\text{card}((A \cap B) \cap (A \cap C)) \\
&= \text{card}(A) + \text{card}(B) + \text{card}(C) - \text{card}(B \cap C) \\
&\quad - (\text{card}(A \cap B) + \text{card}(A \cap C) - \text{card}(A \cap B \cap C)) \\
&= \text{card}(A) + \text{card}(B) + \text{card}(C) \\
&\qquad -\text{card}(A \cap B) - \text{card}(B \cap C) - \text{card}(C \cap A) \\
&\qquad +\text{card}(A \cap B \cap C).
\end{aligned}$$

This gives the Inclusion-Exclusion Formula for three sets. See also figure 2.3.

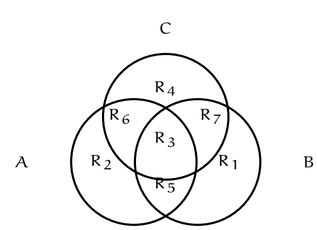


Figure 2.3: Three-set Inclusion-Exclusion

Observe that in the Venn diagram in figure 2.3 there are 8 disjoint regions (the 7 that form $A \cup B \cup C$ and the outside region, devoid of any element belonging to $A \cup B \cup C$).

143 Example How many integers between 1 and 600 inclusive are not divisible by neither 3, nor 5, nor 7?

Solution: Let A_k denote the numbers in $[1; 600]$ which are divisible by $k = 3, 5, 7$. Then

$$\begin{aligned}
|A_3| &= \lfloor \frac{600}{3} \rfloor &= 200, \\
|A_5| &= \lfloor \frac{600}{5} \rfloor &= 120, \\
|A_7| &= \lfloor \frac{600}{7} \rfloor &= 85, \\
|A_{15}| &= \lfloor \frac{600}{15} \rfloor &= 40 \\
|A_{21}| &= \lfloor \frac{600}{21} \rfloor &= 28 \\
|A_{35}| &= \lfloor \frac{600}{35} \rfloor &= 17 \\
|A_{105}| &= \lfloor \frac{600}{105} \rfloor &= 5
\end{aligned}$$

By Inclusion-Exclusion there are $200 + 120 + 85 - 28 - 21 - 17 + 5 = 325$ integers in $[1; 600]$ divisible by at least one of 3, 5, or 7. Those not divisible by these numbers are a total of $600 - 325 = 275$.

144 Example In a group of 30 people, 8 speak English, 12 speak Spanish and 10 speak French. It is known that 5 speak English and Spanish, 5 Spanish and French, and 7 English and French. The number of people speaking all three languages is 3. How many do not speak any of these languages?

Solution: Let A be the set of all English speakers, B the set of Spanish speakers and C the set of French speakers in our group. We fill-up the Venn diagram in figure 2.4 successively. In the intersection of all three we put 8. In the region common to A and B which is not filled up we put $5-2=3$. In the region common to A and C which is not already filled up we put $5-3=2$. In the region common to B and C which is not already filled up, we put $7-3=4$. In the remaining part of A we put $8-2-3-2=1$, in the remaining part of B we put $12-4-3-2=3$, and in the remaining part of C we put $10-2-3-4=1$. Each of the mutually disjoint regions comprise a total of $1+2+3+4+1+2+3=16$ persons. Those outside these three sets are then $30-16=14$.

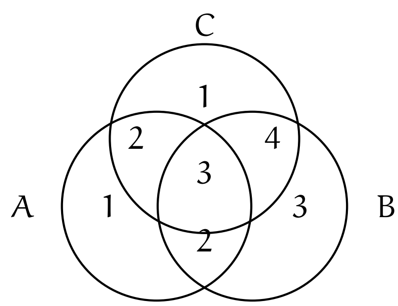


Figure 2.4: Example 144.

145 Example Would you believe a market investigator that reports that of 1000 people, 816 like candy, 723 like ice cream, 645 cake, while 562 like both candy and ice cream, 463 like both candy and cake, 470 both ice cream and cake, while 310 like all three? State your reasons!

Solution: Let C denote the set of people who like candy, I the set of people who like ice cream, and K denote the set of people who like cake. We are given that card$(C)=816$, card$(I)=723$, card$(K)=645$, card$(C\cap I)=562$, card$(C\cap K)=463$, card$(I\cap K)=470$, and card $(C \cap I \cap K) = 310$. By Inclusion-Exclusion we have

$$\begin{aligned}\text{card}(C \cup I \cup K) &= \text{card}(C) + \text{card}(I) + \text{card}(K) \\ &\quad - \text{card}(C \cap I) - \text{card}(C \cap K) - \text{card}(I \cap C) \\ &\quad + \text{card}(C \cap I \cap K) \\ &= 816 + 723 + 645 - 562 - 463 - 470 + 310 \\ &= 999.\end{aligned}$$

The investigator miscounted, or probably did not report one person who may not have liked any of the three things.

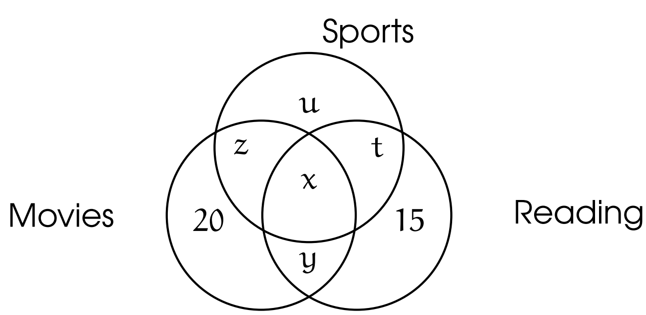


Figure 2.5: Problem 146.

146 Example A survey shews that 90% of high-schoolers in Philadelphia like at least one of the following activities: going to the movies, playing sports, or reading. It is known that 45% like the movies, 48% like sports, and 35% like reading. Also, it is known that 12% like both the movies and reading, 20% like only the movies, and 15% only reading. What percent of high-schoolers like all three activities?

Solution: We make the Venn diagram in as in figure 2.5. From it we gather the following system of equations

$$\begin{array}{ccccccccccccccccc} x & + & y & + & z & & & & & & & + & 20 & = & 45 \\ x & & & + & z & + & t & + & u & & & & & = & 48 \\ x & + & y & & & + & t & & & + & 15 & & & = & 35 \\ x & + & y & & & & & & & & & & & = & 12 \\ x & + & y & + & z & + & t & + & u & + & 15 & + & 20 & = & 90 \end{array}$$

The solution of this system is seen to be $x = 5$, $y = 7$, $z = 13$, $t = 8$, $u = 22$. Thus the percent wanted is 5%.

147 Example An auto insurance company has $10,000$ policyholders. Each policy holder is classified as

- young or old,
- male or female, and
- married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males.

How many of the company's policyholders are young, female, and single?

Solution: Let Y, F, S, M stand for young, female, single, male, respectively, and let Ma stand for married. We have

$$\begin{aligned}\text{card}(Y \cap F \cap S) &= \text{card}(Y \cap F) - \text{card}(Y \cap F \cap Ma) \\ &= \text{card}(Y) - \text{card}(Y \cap M) \\ &\quad -(\text{card}(Y \cap Ma) - \text{card}(Y \cap Ma \cap M)) \\ &= 3000 - 1320 - (1400 - 600) \\ &= 880.\end{aligned}$$

148 Example In *Medieval High* there are forty students. Amongst them, fourteen like Mathematics, sixteen like theology, and eleven like alchemy. It is also known that seven like Mathematics and theology, eight like theology and alchemy and five like Mathematics and alchemy. All three subjects are favoured by four students. How many students like neither Mathematics, nor theology, nor alchemy?

Solution: Let A be the set of students liking Mathematics, B the set of students liking theology, and C be the set of students liking alchemy. We are given that

$$|A| = 14, |B| = 16, |C| = 11, |A \cap B| = 7, |B \cap C| = 8, |A \cap C| = 5,$$

and

$$|A \cap B \cap C| = 4.$$

By the Principle of Inclusion-Exclusion,

$$|\complement A \cap \complement B \cap \complement C| = 40 - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

Substituting the numerical values of these cardinalities

$$40 - 14 - 16 - 11 + 7 + 5 + 8 - 4 = 15.$$

149 Example (AHSME 1991) For a set S, let $n(S)$ denote the number of subsets of S. If A, B, C, are sets for which

$$n(A)+n(B)+n(C)=n(A\cup B\cup C) \text{ and } |A|=|B|=100,$$

then what is the minimum possible value of $|A\cap B\cap C|$?

Solution: A set with k elements has 2^k different subsets. We are given

$$2^{100}+2^{100}+2^{|C|}=2^{|A\cup B\cup C|}.$$

This forces $|C|=101$, as $1+2^{|C|-101}$ is larger than 1 and a power of 2. Hence $|A\cup B\cup C|=102$. Using the Principle Inclusion-Exclusion, since $|A|+|B|+|C|-|A\cup B\cup C|=199$,

$$\begin{aligned}|A\cap B\cap C| &= |A\cap B|+|A\cap C|+|B\cap C|-199\\ &= (|A|+|B|-|A\cup B|)+(|A|+|C|-|A\cup C|)\\ &\quad +(|B|+|C|-|B\cup C|)-199\\ &= 403-|A\cup B|-|A\cup C|-|B\cup C|.\end{aligned}$$

As $A\cup B, A\cup C, B\cup C \subseteq A\cup B\cup C$, the cardinalities of all these sets are ≤ 102. Thus

$$|A\cap B\cap C|=403-|A\cup B|-|A\cup C|-|B\cup C|\geq 403-3\cdot 102=97.$$

The example

$$A=\{1,2,\ldots,100\}, B=\{3,4,\ldots,102\},$$

and

$$C=\{1,2,3,4,5,6,\ldots,101,102\}$$

shews that $|A\cap B\cap C|=|\{4,5,6,\ldots,100\}|=97$ is attainable.

150 Example (Lewis Carroll in *A Tangled Tale.*) In a very hotly fought battle, at least 70% of the combatants lost an eye, at least 75% an ear, at least 80% an arm, and at least 85% a leg. What can be said about the percentage who lost all four members?

Solution: Let A denote the set of those who lost an eye, B denote those who lost an ear, C denote those who lost an arm and D denote those losing a leg. Suppose there are n combatants. Then

$$\begin{aligned} n &\geq \text{card}\,(A \cup B) \\ &= \text{card}\,(A) + \text{card}\,(B) - \text{card}\,(A \cap B) \\ &= .7n + .75n - \text{card}\,(A \cap B)\,, \end{aligned}$$

$$\begin{aligned} n &\geq \text{card}\,(C \cup D) \\ &= \text{card}\,(C) + \text{card}\,(D) - \text{card}\,(C \cap D) \\ &= .8n + .85n - \text{card}\,(C \cap D)\,. \end{aligned}$$

This gives

$$\text{card}\,(A \cap B) \geq .45n,$$

$$\text{card}\,(C \cap D) \geq .65n.$$

This means that

$$\begin{aligned} n &\geq \text{card}\,((A \cap B) \cup (C \cap D)) \\ &= \text{card}\,(A \cap B) + \text{card}\,(C \cap D) - \text{card}\,(A \cap B \cap C \cap D) \\ &\geq .45n + .65n - \text{card}\,(A \cap B \cap C \cap D)\,, \end{aligned}$$

whence

$$\text{card}\,(A \cap B \cap C \cap D) \geq .45 + .65n - n = .1n.$$

This means that at least 10% of the combatants lost all four members.

Homework

151 Problem Let N be a positive integer, and let a, b be positive integers which are relatively prime, that is, which do not share a prime factor in common. Consider the set of positive integers $\{1, 2, 3, \ldots, N\}$.

❶ How many are divisible by a?

❷ How many are divisible by b?

❸ How many are divisible by ab?

❹ How many are divisible by at least one member of the set $\{a, b\}$?

❺ How many are divisible by no member of the set $\{a, b\}$?

❻ How many are divisible by exactly one member of the set $\{a, b\}$?

152 Problem Consider the set of the first 100 positive integers:

$$A = \{1, 2, 3, \ldots, 100\}.$$

❶ How many are divisible by 2?

❷ How many are divisible by 3?

❸ How many are divisible by 7?

❹ How many are divisible by 6?

❺ How many are divisible by 14?

❻ How many are divisible by 21?

❼ How many are divisible by 42?

❽ How many are relatively prime to 42?

❾ How many are divisible by 2 and 3 but not by 7?

➓ How many are divisible by exactly one of 2, 3 and 7?

153 Problem A survey of a groups viewing habits over the last year revealed the following information:

❶ 28% watched gymnastics

❷ 29% watched baseball

❸ 19% watched soccer

❹ 14% watched gymnastics and baseball

❺ 12% watched baseball and soccer

❻ 10% watched gymnastics and soccer

❼ 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

154 Problem Out of 40 children, 30 can swim, 27 can play chess, and only 5 can do neither. How many children can swim and play chess?

Answer: 22

155 Problem Among the school seniors top marks were received by 48 students in Mathematics, 37 in Physics, 42 in Chemistry, 75 in Mathematics and Physics, 76 in Mathematics and Chemistry, 66 in Physics and Chemistry, and 4 students in all three subjects. How many students obtained more than one excellent mark? How many of them obtained only one excellent mark?

Answer: $94; 65$

156 Problem How many positive integers less than a million are neither perfect squares, perfect cubes, perfect fourth powers?

157 Problem (AHSME 1988) X, Y, and Z are pairwise disjoint sets of people. The average ages of people in the sets X, Y, Z, $X \cup Y$, $X \cup Y$, and $Y \cup Z$ are given below:

Set	X	Y	Z	$X \cup Y$	$X \cup Z$	$Y \cup Z$
Average Age	37	23	41	29	39.5	33

What is the average age of the people in the set $X \cup Y \cup Z$?

Answer: 34

158 Problem Each of the students in the maths class twice attended a concert. It is known that $25, 12$, and 23 students attended concerts A, B, and C respectively. How many students are there in the maths class? How many of them went to concerts A and B, B and C, or B and C?

Answer: $30; 7; 5; 18$

159 Problem The films A, B, and C were shewn in the cinema for a week. Out of 40 students (each of which saw either all the three films, or one of them, 13 students saw film A, 16 students saw film B, and 19 students saw film C. How many students saw all three films?

Answer: 4

2.2 The Product Rule

160 Rule (Product Rule: Cartesian Product Form) Let $A_1, A_2, \ldots, A_k$, be finite sets with $\text{card}\,(A)_i = n_i$. Then the cardinality of their cartesian product is

$$\begin{aligned}\text{card}\,(A_1 \times A_2 \times \cdots \times A_k) &= \text{card}\,(A_1) \cdot \text{card}\,(A_2) \cdots \text{card}\,(A_k) \\ &= n_1 n_2 \cdots n_k.\end{aligned}$$

We may formulate the following alternative form of the multiplication rule.

161 Rule (Product Rule: Sequential Form) Suppose that an experiment E can be performed in k stages: E_1 first, E_2 second, ..., E_k last. Suppose moreover that E_i can be done in n_i different ways, and that the number of ways of performing E_i is not influenced by any predecessors $E_1, E_2, \ldots, E_{i-1}$. Then E_1 **and** E_2 **and** ...**and** E_k can occur simultaneously in $n_1 n_2 \cdots n_k$ ways.

162 Example In a group of 8 men and 9 women we can pick one man **and** one woman in $8 \cdot 9 = 72$ ways. Notice that we are choosing two persons.

163 Example A red die and a blue die are tossed. In how many ways can they land?

Solution: If we view the outcomes as an ordered pair (r, b) then by the multiplication principle we have the $6 \cdot 6 = 36$ possible outcomes

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

The red die can land in any of 6 ways,

$$\begin{array}{|c|c|}\hline 6 & \quad \\ \hline\end{array}$$

and also, the blue die may land in any of 6 ways

$$\begin{array}{|c|c|}\hline 6 & 6 \\ \hline\end{array}.$$

164 Example There are $9 \cdot 10 \cdot 10 = 900$ positive 3-digit integers:

$$100, 101, 102, \ldots, 998, 999.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\begin{array}{|c|c|c|}\hline 9 & \quad & \quad \\ \hline\end{array}.$$

There are 10 choices for the second digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & \quad \\ \hline\end{array},$$

and also 10 choices for the last digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 10 \\ \hline\end{array}.$$

165 Example There are $9 \cdot 10 \cdot 5 = 450$ even positive 3-digit integers:

$$100, 102, 104, \ldots, 996, 998.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\begin{array}{|c|c|c|}\hline 9 & \quad & \quad \\ \hline\end{array}.$$

There are 10 choices for the second digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & \quad \\ \hline\end{array}.$$

Since the integer must be even, the last digit must be one of the 5 choices $\{0, 2, 4, 6, 8\}$

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 5 \\ \hline\end{array}.$$

166 Definition A *palindromic integer* or *palindrome* is a positive integer whose decimal expansion is symmetric and that is not divisible by 10. In other words, one reads the same integer backwards or forwards.

167 Example The following integers are all palindromes:

$$1, 8, 11, 99, 101, 131, 999, 1234321, 9987899.$$

168 Example There are $9 \cdot 10 \cdot 10 = 900$ palindromes of 5 digits.

$$\begin{array}{|c|c|c|c|c|}\hline 9 & \quad & \quad & \quad & \quad \\ \hline\end{array}.$$

Once the leftmost digit is chosen, the last digit must be identical to it, so we have

$$\begin{array}{|c|c|c|c|c|}\hline 9 & \quad & \quad & \quad & 1 \\ \hline\end{array}.$$

There are 10 choices for the second digit from the left

$$\begin{array}{|c|c|c|c|c|}\hline 9 & 10 & \quad & \quad & 1 \\ \hline\end{array}.$$

Once this digit is chosen, the second digit from the right must be identical to it, so we have only 1 choice for it,

9	10		1	1

.

Finally, there are 10 choices for the third digit from the right,

9	10	10	1	1

,

which give us 900 palindromes of 3-digits.

169 Example Out of nine different pairs of shoes, in how many ways could I choose a right shoe and a left shoe, which should not form a pair?

Solution: I can choose a right shoe in any of nine ways, once this has been done, I can choose a non-matching left shoe in eight ways, and thus I have 72 choices.
Aliter: I can choose any pair in $9 \times 9 = 81$ ways. Of these, 9 are matching pairs, so the number of non-matching pairs is $81 - 9 = 72$.

170 Example A license plate is to be made according to the following provision: it has four characters, the first two characters can be any letter of the English alphabet and the last two characters can be any digit. One is allowed to repeat letters and digits. How many different license plates can be made?

Solution: The number of different license plates is the number of different four-tuples (Letter $_1$, Letter $_2$, Digit $_1$, Digit $_2$). The first letter can be chosen in 26 ways, and so we have

26			

.

The second letter can be chosen in any of 26 ways:

26	26		

.

The first digit can be chosen in 10 ways:

26	26	10	

.

Finally, the last digit can be chosen in 10 ways:

26	26	10	10

.

By the multiplication principle, the number of different four-tuples is $26 \cdot 26 \cdot 10 \cdot 10 = 67600$.

171 Example In the example 170, how many different license plates can you make if (i) you may repeat letters but not digits?, (ii) you may repeat digits but not letters?, (iii) you may repeat neither letters nor digits?

Solution: (i) In this case we have a grid like

26	26	10	9

,

since after a digit has been used for the third position, it cannot be used again. Thus this can be done in $26 \cdot 26 \cdot 10 \cdot 9 = 60840$ ways.
(ii) In this case we have a grid like

26	25	10	10

,

since after a letter has been used for the first position, it cannot be used again. Thus this can be done in $26 \cdot 25 \cdot 10 \cdot 10 = 65000$ ways.
(iii) After a similar reasoning, we obtain a grid like

26	25	10	9

.

Thus this can be done in $26 \cdot 25 \cdot 10 \cdot 9 = 58500$ ways.

172 Example How many distinct four-letter words can be made with the letters of the set $\{c, i, k, t\}$

❶ if the letters are not to be repeated?

❷ if the letters can be repeated?

Solution:

❶ The first letter can be one of any 4. After choosing the first letter, we have 3 choices for the second letter, etc.. The total number of words is thus $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

❷ The first letter can be one of any 4. Since we are allowed repetitions, the second letter can also be one of any 4, etc.. The total number of words so formed is thus $4^4 = 256$.

173 Example How many distinct six-digit numbers that are multiples of 5 can be formed from the list of digits $\{1, 2, 3, 4, 5, 6\}$ if we allow repetition?

Solution: The last digit must perforce be 5. The other five digits can be filled with any of the six digits on the list: the total number is thus 6^5.

174 Example Telephone numbers in *Land of the Flying Camels* have 7 digits, and the only digits available are $\{0, 1, 2, 3, 4, 5, 7, 8\}$. No telephone number may begin in 0, 1 or 5. Find the number of telephone numbers possible that meet the following criteria:

❶ You may repeat all digits.

❷ You may not repeat any of the digits.

❸ You may repeat the digits, but the phone number must be even.

❹ You may repeat the digits, but the phone number must be odd.

❺ You may not repeat the digits and the phone numbers must be odd.

Solution:

❶ This is $5 \cdot 8^6 = 1310720$.

❷ This is $5 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 25200$.

❸ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❹ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❺ We condition on the last digit. If the last digit were 1 or 5 then we would have 5 choices for the first digit, and so we would have

$$5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 7200$$

phone numbers. If the last digit were either 3 or 7, then we would have 4 choices for the last digit and so we would have

$$4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 5760$$

phone numbers. Thus the total number of phone numbers is

$$7200 + 5760 = 12960.$$

175 Example (AIME 1993) How many even integers between 4000 and 7000 have four different digits?

Solution: We condition on the first digit, which can be $4, 5$, or 6. If the number starts with 4, in order to satisfy the conditions of the problem, we must choose the last digit from the set $\{0, 2, 6, 8\}$. Thus we have four choices for the last digit. Once this last digit is chosen, we have 8 choices for the penultimate digit and 7 choices for the antepenultimate digit. There are thus $4 \times 8 \times 7 = 224$ even numbers which have their digits distinct and start with a 4. Similarly, there are 224 even numbers will all digits distinct and starting with a 6. When they start with a 5, we have 5 choices for the last digit, 8 for the penultimate and 7 for the antepenultimate. This gives $5 \times 8 \times 7 = 280$ ways. The total number is thus $224 + 224 + 280 = 728$.

176 Example How many positive divisors does 300 have?

Solution: We have $300 = 3 \cdot 2^2 5^2$. Thus every factor of 300 is of the form $3^a 2^b 5^c$, where $0 \le a \le 1$, $0 \le b \le 2$, and $0 \le c \le 2$. Thus there are 2 choices for a, 3 for b and 3 for c. This gives $2 \cdot 3 \cdot 3 = 18$ positive divisors.

177 Example How many positive divisors does $2^8 3^9 5^2$ have? What is the sum of these divisors?

Solution: We will assume that the positive integers may be factorised in a unique manner as the product of primes. Expanding the product

$$(1 + 2 + 2^2 + \cdots + 2^8)(1 + 3 + 3^2 + \cdots + 3^9)(1 + 5 + 5^2)$$

each factor of $2^8 3^9 5^2$ appears and only the factors of this number appear. There are then, as many factors as terms in this product. This means that there are $(1 + 8)(1 + 9)(1 + 3) = 320$ factors.

The sum of the divisors of this number may be obtained by adding up each geometric series in parentheses. The desired sum is then

$$\frac{2^9 - 1}{2 - 1} \cdot \frac{3^{10} - 1}{3 - 1} \cdot \frac{5^3 - 1}{5 - 1} = 467689684.$$

A similar argument gives the following. Let $p_1, p_2, \ldots, p_k$ be different primes. Then the integer

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

has

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

positive divisors. Also, if $\sigma(n)$ denotes the sum of all positive divisors of n, then

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

178 Example How many factors of 2^{95} are larger than $1,000,000$?

Solution: The 96 factors of 2^{95} are $1, 2, 2^2, \ldots, 2^{95}$. Observe that $2^{10} = 1024$ and so $2^{20} = 1048576$. Hence

$$2^{19} = 524288 < 1000000 < 1048576 = 2^{20}.$$

The factors greater than $1,000,000$ are thus $2^{20}, 2^{21}, \ldots 2^{95}$. This makes for $96 - 20 = 76$ factors.

179 Example How many palindromes of 5 digits are even?

Solution: A five digit even palindrome has the form $ABCBA$, where A belongs to $\{2,4,6,8\}$, and B, C belong to $\{0,1,2,3,4,5,6,7,8,9\}$. Thus there are 4 choices for the first digit, 10 for the second, and 10 for the third. Once these digits are chosen, the palindrome is completely determined. Therefore, there are $4 \times 10 \times 10 = 400$ even palindromes of 5 digits.

180 Example A multiple-choice test consists of 20 questions, each one with 4 choices. There are 4 ways of answering the first question, 4 ways of answering the second question, etc., hence there are $4^{20} = 1099511627776$ ways of answering the exam.

181 Example Generalising example 164, let $n \geq 1$ be an integer. There are $9 \cdot 10^{n-1}$ positive integers with n digits. For the leftmost digit cannot be 0 and so we have only the nine choices

$$\{1,2,3,4,5,6,7,8,9\}$$

for this digit. The other $n - 1$ digits can be filled out in 10 ways, and so there are

$$9 \cdot \underbrace{10 \cdots 10}_{n-1 \text{ } 10's} = 9 \cdot 10^{n-1}.$$

182 Example Generalising example 165, let $n \geq 2$ be an integer. There are $45 \cdot 10^{n-2}$ even positive integers with n digits. For the leftmost digit cannot be 0 and so we have only the nine choices

$$\{1,2,3,4,5,6,7,8,9\}$$

for this digit. If the integer is going to be even, the last digit can be only one of the five $\{0,2,4,6,8\}$. The other $n - 2$ digits can be filled out in 10 ways, and so there are

$$9 \cdot \underbrace{10 \cdots 10}_{n-2 \text{ } 10's} \cdot 5 = 45 \cdot 10^{n-2}.$$

183 Example How many n-digit numbers do not have the digit 0?

Solution: One can choose the last digit in 9 ways, one can choose the penultimate digit in 9 ways, etc. and one can choose the second digit in 9 ways, and finally one can choose the first digit in 9 ways. The total number of ways is thus 9^n.

184 Example How many paths consisting of a sequence of horizontal and/or vertical line segments, each segment connecting a pair of adjacent letters in figure 2.6 spell BIPOLAR?

B
B I B
B I P I B
B I P O P I B
B I P O L O P I B
B I P O L A L O P I B
B I P O L A R A L O P I B

Figure 2.6: Problem 184.

Solution: Split the diagram, as in figure 2.7. Since every required path must use the R, we count paths starting from R and reaching up to a B. Since there are six more rows that we can travel to, and since at each stage we can go either up or left, we have $2^6 = 64$ paths. The other half of the figure will provide 64 more paths. Since the middle column is shared by both halves, we have a total of $64 + 64 - 1 = 127$ paths.

```
            B
          B I
        B I P
      B I P O
    B I P O L
  B I P O L A
B I P O L A R
```

Figure 2.7: Problem 184.

185 Example How many n-digit nonnegative integers do not contain the digit 5?

Answer: 9 1-digit numbers and $8 \cdot 9^{n-1}$ n-digit numbers $n \geq 2$.

We now prove that if a set A has n elements, then it has 2^n subsets. To motivate the proof, consider the set $\{a, b, c\}$. To each element we attach a binary code of length 3. We write 0 if a particular element is not in the set and 1 if it is. Thus we have the following associations:

$$\varnothing \leftrightarrow 000,$$

$$\{a\} \leftrightarrow 100,$$
$$\{b\} \leftrightarrow 010,$$
$$\{c\} \leftrightarrow 001,$$
$$\{a, b\} \leftrightarrow 110,$$
$$\{a, c\} \leftrightarrow 101,$$
$$\{b, c\} \leftrightarrow 011,$$
$$\{a, b, c\} \leftrightarrow 111.$$

Thus there is a one-to-one correspondence between the subsets of a finite set of 3 elements and binary sequences of length 3.

186 Theorem (Cardinality of the Power Set) Let A be a finite set with card $(A) = n$. Then A has 2^n subsets, that is, card $(\mathscr{P}(A)) = 2^n$.

Proof We attach a binary code to each element of the subset, 1 if the element is in the subset and 0 if the element is not in the subset. The total number of subsets is the total number of such binary codes, and there are 2^n in number. ❑

187 Example Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

Solution: There are 589 such values. The easiest way to see this is to observe that there is a bijection between the divisors of n^2 which are $> n$ and those $< n$. For if $n^2 = ab$, with $a > n$, then $b < n$, because otherwise $n^2 = ab > n \cdot n = n^2$, a contradiction. Also, there is exactly one decomposition $n^2 = n \cdot n$. Thus the desired number is

$$\lfloor \frac{d(n^2)}{2} \rfloor + 1 - d(n) = \lfloor \frac{(63)(39)}{2} \rfloor + 1 - (32)(20) = 589.$$

188 Example Let $n \geq 3$. Find the number of n-digit ternary sequences that contain at least one 0, one 1 and one 2.

Solution: The total number of sequences is 3^n. There are 2^n sequences that contain no 0, 1 or 2. There is only one sequence that contains only 1's, one that contains only 2's, and one that contains only 0's. Obviously, there is no ternary sequence that contains no 0's or 1's or 2's. By the Principle of Inclusion-Exclusion, the number required is

$$3^n - (2^n + 2^n + 2^n) + (1 + 1 + 1) = 3^n - 3 \cdot 2^n + 3.$$

189 Example In how many ways can one decompose the set

$$\{1, 2, 3, \ldots, 100\}$$

into subsets A, B, C satisfying

$$A \cup B \cup C = \{1, 2, 3, \ldots, 100\} \quad \text{and} \quad A \cap B \cap C = \varnothing$$

Solution: The conditions of the problem stipulate that both the region outside the circles in diagram 2.3 and R_3 will be empty. We are thus left with 6 regions to distribute 100 numbers. To each of the 100 numbers we may thus assign one of 6 labels. The number of sets thus required is 6^{100}.

Homework

190 Problem A true or false exam has ten questions. How many possible answer keys are there?

Answer: 1024

191 Problem In how many ways can the following prizes be given away to a class of twenty boys: first and second Classical, first and second Mathematical, first Science, and first French?

Answer: 57760000

192 Problem Under old hardware, a certain programme accepted passwords of the form

$$eell$$

where $e \in \{0, 2, 4, 6, 8\}$ and $l \in \{a, b, c, d, u, v, w, x, y, z\}$. The hardware was changed and now the software accepts passwords of the form

$$eeelll.$$

How many more passwords of the latter kind are there than of the former kind?

Answer: 122500

193 Problem An alphabet consists of the **five** consonants {p, v, t, s, k} and the **three** vowels {a, e, o}. A license plate is to be made using **four** letters of this alphabet.

❶ How many letters does this alphabet have?

❷ If a license plate is of the form $CCVV$ where C denotes a consonant and V denotes a vowel, how many possible license plates are there, assuming that you may repeat both consonants and vowels?

❸ If a license plate is of the form $CCVV$ where C denotes a consonant and V denotes a vowel, how many possible license plates are there, assuming that you may repeat consonants but not vowels?

❹ If a license plate is of the form $CCVV$ where C denotes a consonant and V denotes a vowel, how many possible license plates are there, assuming that you may repeat vowels but not consonants?

❺ If a license plate is of the form $LLLL$ where L denotes any letter of the alphabet, how many possible license plates are there, assuming that you may not repeat letters?

194 Problem A man lives within reach of three boys' schools and four girls' schools. In how many ways can he send his three sons and two daughters to school?

Answer: 432

195 Problem How many 5-lettered words can be made out of 26 letters, repetitions allowed, but not consecutive repetitions (that is, a letter may not follow itself in the same word)?

Answer: 10156250

196 Problem There are m different roads from town A to town B. In how many ways can Dwayne travel from town A to town B and back if (a) he may come back the way he went?, (b) he must use a different road of return?

Answer: m^2, $m(m-1)$

197 Problem How many positive divisors does 360 have? How many are even? How many are odd? How many are perfect squares?

198 Problem (AHSME 1988) At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First # 5 bowls #4. The loser receives the 5th prize and the winner bowls # 3 in another game. The loser of this game receives the 4th prize and the winner bowls # 2. The loser of this game receives the 3rd prize and the winner bowls # 1. The loser of this game receives the 2nd prize and the winner the 1st prize. In how many orders can bowlers #1 through #5 receive the prizes?

Answer: 16

199 Problem A square chessboard has 16 squares (4 rows and 4 columns). One puts 4 checkers in such a way that only one checker can be put in a square. Determine the number of ways of putting these checkers if

❶ there must be exactly one checker per row and column.

❷ there must be exactly one column without a checker.

❸ there must be at least one column without a checker.

Answer: $24, 1152, 1564$

200 Problem The password of the anti-theft device of a car is a four digit number, where one can use any digit in the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

A.
- ❶ How many such passwords are possible?
- ❷ How many of the passwords have all their digits distinct?

B. After an electrical failure, the owner must reintroduce the password in order to deactivate the anti-theft device. He knows that the four digits of the code are $2, 0, 0, 3$ but does not recall the order.

- ❶ How many such passwords are possible using only these digits?
- ❷ If the first attempt at the password fails, the owner must wait two minutes before a second attempt, if the second attempt fails he must wait four minutes before a third attempt, if the third attempt fails he must wait eight minutes before a fourth attempt, etc. (the time doubles from one attempt to the next). How many passwords can the owner attempt in a period of 24 hours?

Answer: A. 10000, 5040, B. 12 , 10

201 Problem The number 3 can be expressed as a sum of one or more positive integers in four ways, namely, as 3, $1 + 2$, $2 + 1$, and $1 + 1 + 1$. Shew that any positive integer n can be so expressed in 2^{n-1} ways.

2.3 The Sum Rule

202 Rule (Sum Rule: Set Union Form) Let $A_1, A_2, \ldots, A_k$, be finite sets with card $(A)_i = n_i$, and assume that they are pairwise disjoint, that is, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then

$$\text{card}\,(A_1 \cup A_2 \cup \cdots \cup A_k) = n_1 + n_2 + \cdots n_k.$$

203 Definition Two events are said to be *mutually exclusive* if the occurrence of one prevents the occurrence of the other, that is, they cannot occur simultaneously.

The sum rule can be reformulated as follows.

204 Rule (Sum Rule: Disjunctive Form) Let $E_1, E_2, \ldots, E_k$, be pairwise mutually exclusive events. If E_i can occur in n_i ways, then either E_1 **or** E_2 **or**, $\ldots$, **or** E_k can occur in

$$n_1 + n_2 + \cdots n_k$$

ways.

*Notice that the "**or**" here is exclusive.*

205 Example In a group of 8 men and 9 women we can pick one man **or** one woman in $8 + 9 = 17$ ways. Notice that we are choosing one person.

206 Example There are five Golden retrievers, six Irish setters, and eight Poodles at the pound. How many ways can two dogs be chosen if they are not the same kind.

Solution: We choose: a Golden retriever **and** an Irish setter **or** a Golden retriever **and** a Poodle **or** an Irish setter **and** a Poodle.

One Golden retriever and one Irish setter can be chosen in $5 \cdot 6 = 30$ ways; one Golden retriever and one Poodle can be chosen in $5 \cdot 8 = 40$ ways; one Irish setter and one Poodle can be chosen in $6 \cdot 8 = 48$ ways. By the sum rule, there are $30 + 40 + 48 = 118$ combinations.

207 Example To write a book 1890 digits were utilised. How many pages does the book have?

Solution: A total of

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits are used to write pages 1 to 99, inclusive. We have of $1890 - 189 = 1701$ digits at our disposition which is enough for $1701/3 = 567$ extra pages (starting from page 100). The book has $99 + 567 = 666$ pages.

208 Example All the positive integers with initial digit 2 are written in succession:

$$2, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 200, 201, \ldots,$$

Find the 1978-th digit written.

Solution: There is 1 such number with 1 digit, 10 such numbers with 2 digits, 100 with three digits, 1000 with four digits, etc. Starting with 2 and finishing with 299 we have used $1 \cdot 1 + 2 \cdot 10 + 3 \cdot 100 = 321$ digits. We need $1978 - 321 = 1657$ more digits from among the 4-digit integers starting with 2. Now $\lfloor \frac{1657}{4} \rfloor = 414$, so we look at the 414th 4-digit integer starting with 2, namely, at 2413. Since the 3 in 2413 constitutes the $321 + 4 \cdot 414 = 1977$-th digit used, the 1978-th digit must be the 2 starting 2414.

209 Example The sequence of palindromes, starting with 1 is written in ascending order

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots$$

Find the 1984-th positive palindrome.

Solution: It is easy to see that there are 9 palindromes of 1 digit, 9 palindromes with two digits, 90 with three digits, 90 with 4 digits, 900 with 5 digits and 900 with 6 digits. The last palindrome with 6 digits, 999999, constitutes the $9+9+90+90+900+900 = 1998$th palindrome. Hence, the 1997th palindrome is 998899, the 1996th palindrome is 997799, the 1995th palindrome is 996699, the 1994th is 995599, etc., until we find the 1984th palindrome to be 985589.

210 Example Find the sum of all odd 5-digit palindromes.

Solution: By example 168 there are 900 5-digit palindromes, and by example 179, there are $4 \times 10 \times 10 = 400$ even palindromes of five digits. Thus there are $900 - 400 = 500$ odd palindromes of five digits. Observe that each pair below has the same sum

$$110000 = 10001 + 99999 = 10101 + 99899 = \cdots .$$

Since there are 250 such pairs, the total sum is thus

$$110000 \times 250 = 27500000.$$

211 Example The integers from 1 to 1000 are written in succession. Find the sum of all the digits.

Solution: When writing the integers from 000 to 999 (with three digits), $3 \times 1000 = 3000$ digits are used. Each of the 10 digits is used an equal number of times, so each digit is used 300 times. The the sum of the digits in the interval 000 to 999 is thus

$$(0+1+2+3+4+5+6+7+8+9)(300) = 13500.$$

Therefore, the sum of the digits when writing the integers from 000 to 1000 is $13500 + 1 = 13501$.

212 Example How many 4-digit integers can be formed with the set of digits $\{0, 1, 2, 3, 4, 5\}$ such that no digit is repeated and the resulting integer is a multiple of 3?

Solution: The integers desired have the form $D_1D_2D_3D_4$ with $D_1 \neq 0$. Under the stipulated constraints, we must have

$$D_1 + D_2 + D_3 + D_4 \in \{6, 9, 12\}.$$

We thus consider three cases.

Case I: $D_1 + D_2 + D_3 + D_4 = 6$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 1, 2, 3, 4\}, D_1 \neq 0$. There are then 3 choices for D_1. After D_1 is chosen, D_2 can be chosen in 3 ways, D_3 in 2 ways, and D_1 in 1 way. There are thus $3 \times 3 \times 2 \times 1 = 3 \cdot 3! = 18$ integers satisfying case I.

Case II: $D_1 + D_2 + D_3 + D_4 = 9$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 2, 3, 4\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{0, 1, 3, 5\}, D_1 \neq 0$. Like before, there are $3 \cdot 3! = 18$ numbers in each possibility, thus we have $2 \times 18 = 36$ numbers in case II.

Case III: $D_1 + D_2 + D_3 + D_4 = 12$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 3, 4, 5\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{1, 2, 4, 5\}$. In the first possibility there are $3 \cdot 3! = 18$ numbers, and in the second there are $4! = 24$. Thus we have $18 + 24 = 42$ numbers in case III.

The desired number is finally $18 + 36 + 42 = 96$.

213 Example Let S be the set of all natural numbers whose digits are chosen from the set $\{1, 3, 5, 7\}$ such that no digits are repeated. Find the sum of the elements of S.

Solution: First observe that $1 + 7 = 3 + 5 = 8$. The numbers formed have either one, two, three or four digits. The sum of the numbers of 1 digit is clearly $1 + 7 + 3 + 5 = 16$.

There are $4 \times 3 = 12$ numbers formed using 2 digits, and hence 6 pairs adding to 8 in the units and the tens. The sum of the 2 digits formed is $6((8)(10) + 8) = 6 \times 88 = 528$.

There are $4 \times 3 \times 2 = 24$ numbers formed using 3 digits, and hence 12 pairs adding to 8 in the units, the tens, and the hundreds. The sum of the 3 digits formed is $12(8(100) + (8)(10) + 8) = 12 \times 888 = 10656$.

There are $4 \times 3 \times 2 \cdot 1 = 24$ numbers formed using 4 digits, and hence 12 pairs adding to 8 in the units, the tens the hundreds, and the thousands. The sum of the 4 digits formed is $12(8(1000) + 8(100) + (8)(10) + 8) = 12 \times 8888 = 106656$.

The desired sum is finally

$$16 + 528 + 10656 + 106656 = 117856.$$

214 Example Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, \ldots, 50\}$ such that

❶ $|a - b| = 5$

❷ $|a - b| \leq 5$.

Solution:

❶ We find the pairs

$$\{1, 6\}, \{2, 7\}, \{3, 8\}, \ldots, \{45, 50\},$$

so there are 45 in total. (Note: the pair $\{a, b\}$ is indistinguishable from the pair $\{b, a\}$.

❷ If $|a - b| = 1$, then we have

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{49, 50\},$$

or 49 pairs. If $|a - b| = 2$, then we have

$$\{1, 3\}, \{2, 4\}, \{3, 5\}, \ldots, \{48, 50\},$$

or 48 pairs. If $|a - b| = 3$, then we have

$$\{1, 4\}, \{2, 5\}, \{3, 6\}, \ldots, \{47, 50\},$$

or 47 pairs. If $|a - b| = 4$, then we have

$$\{1, 5\}, \{2, 6\}, \{3, 7\}, \ldots, \{46, 50\},$$

or 46 pairs. If $|a-b|=5$, then we have

$$\{1,6\},\{2,7\},\{3,8\},\ldots,\{45,50\},$$

or 45 pairs.

The total required is thus

$$49+48+47+46+45=235.$$

215 Example (AIME 1994) Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S=p(1)+p(2)+\cdots+p(999).$$

Find S.

Solution: If $x=0$, put $m(x)=1$, otherwise put $m(x)=x$. We use three digits to label all the integers, from 000 to 999 If a,b,c are digits, then clearly $p(100a+10b+c)=m(a)m(b)m(c)$. Thus

$$\begin{aligned} p(000)+p(001)+p(002)+\cdots+p(999) &= m(0)m(0)m(0)+m(0)m(0)m(1) \\ &+ m(0)m(0)m(2)+\cdots \\ &\quad +m(9)m(9)m(9) \\ &= (m(0)+m(1)+\cdots+m(9))^3 \\ &= (1+1+2+\cdots+9)^3 \\ &= 46^3 \\ &= 97336. \end{aligned}$$

Hence

$$\begin{aligned} S &= p(001) + p(002) + \cdots + p(999) \\ &= 97336 - p(000) \\ &= 97336 - m(0)m(0)m(0) \\ &= 97335. \end{aligned}$$

Homework

216 Problem How many different sums can be thrown with two dice, the faces of each die being numbered $0, 1, 3, 7, 15, 31$?

Answer: 21

217 Problem How many different sums can be thrown with three dice, the faces of each die being numbered $1, 4, 13, 40, 121, 364$?

Answer: 56

218 Problem How many two or three letter initials for people are available if at least one of the letters must be a D and one does not allow repetitions? What if one allows repetitions?

Answer: $1925, 2002$

219 Problem How many positive integers have all their digits distinct?

Answer:

$$\begin{aligned} 9+9\cdot 9 & \\ +9\cdot 9\cdot 8+9\cdot 9\cdot 8\cdot 7 & \\ +9\cdot 9\cdot 8\cdot 7\cdot 6+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5 & \\ +9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3 & \\ +9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 &= 8877690 \end{aligned}$$

220 Problem The Morse code consists of points and dashes. How many letters can be in the Morse code if no letter contains more than four signs, but all must have at least one?

Answer: 30.

221 Problem An $n\times n\times n$ wooden cube is painted blue and then cut into n^3 $1\times 1\times 1$ cubes. How many cubes (a) are painted on exactly three sides, (b) are painted in exactly two sides, (c) are painted in exactly one side, (d) are not painted?

Answer: 8; $12(n-2)$; $6(n-2)^2$; $(n-2)^3$
Comment: This proves that $n^3=(n-2)^3+6(n-2)^2+12(n-2)+8$.

222 Problem (AHSME 1998) Call a 7-digit telephone number $d_1d_2d_3-d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ or possibly both. Assuming that each d_i can be any of the ten decimal digits $0,1,2,\ldots,9$, find the number of different memorable telephone numbers.

Answer: 19990

223 Problem Three-digit numbers are made using the digits $\{1,3,7,8,9\}$.

❶ How many of these integers are there?

❷ How many are even?

❸ How many are palindromes?

❹ How many are divisible by 3?

224 Problem (AHSME 1989) Five people are sitting at a round table. Let $f \geq 0$ be the number of people sitting next to at least one female, and let $m \geq 0$ be the number of people sitting next to at least one male. Find the number of possible ordered pairs (f, m).

Answer: 8.

225 Problem How many integers less than 10000 can be made with the eight digits $0, 1, 2, 3, 4, 5, 6, 7?$

Answer: 4095.

226 Problem (ARML 1999) In how many ways can one arrange the numbers $21, 31, 41, 51, 61, 71$, and 81 such that the sum of every four consecutive numbers is divisible by 3?

Answer: 144

2.4 Permutations without Repetitions

227 Definition We define the symbol ! (factorial), as follows: $0! = 1$, and for integer $n \geq 1$,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$n!$ is read n *factorial*.

228 Example

$$\begin{aligned} 1! &= 1, \\ 2! &= 1 \cdot 2 = 2, \\ 3! &= 1 \cdot 2 \cdot 3 = 6 \\ 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24 \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120. \end{aligned}$$

229 Example

$$\begin{aligned} \frac{7!}{4!} &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210, \\ \frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1), \\ \frac{(n-2)!}{(n+1)!} &= \frac{(n-2)!}{(n+1)(n)(n-1)(n-2)!} = \frac{1}{(n+1)(n)(n-1)}. \end{aligned}$$

230 Definition Let $x_1, x_2, \ldots, x_n$ be n distinct objects. A *permutation* of these objects is simply a rearrangement of them.

231 Example There are 24 permutations of the letters in MATH, namely

MATH	MAHT	MTAH	MTHA	MHTA	MHAT
AMTH	AMHT	ATMH	ATHM	AHTM	AHMT
TAMH	TAHM	TMAH	TMHA	THMA	THAM
HATM	HAMT	HTAM	HTMA	HMTA	HMAT

232 Theorem Let $x_1, x_2, \ldots, x_n$ be n distinct objects. Then there are $n!$ permutations of them.

Proof The first position can be chosen in n ways, the second object in $n-1$ ways, the third in $n-2$, etc. This gives

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!.$$

233 Example The number of permutations of the letters of the word RETICULA is $8! = 40320$.

234 Example A bookshelf contains 5 German books, 7 Spanish books and 8 French books. Each book is different from one another.

❶ How many different arrangements can be done of these books?

❷ How many different arrangements can be done of these books if books of each language must be next to each other?

❸ How many different arrangements can be done of these books if all the French books must be next to each other?

❹ How many different arrangements can be done of these books if no two French books must be next to each other?

Solution:

❶ We are permuting $5+7+8=20$ objects. Thus the number of arrangements sought is $20! = 2432902008176640000$.

❷ "Glue" the books by language, this will assure that books of the same language are together. We permute the 3 languages in $3!$ ways. We permute the German books in $5!$ ways, the Spanish books in $7!$ ways and the French books in $8!$ ways. Hence the total number of ways is $3!5!7!8! = 146313216000$.

❸ Align the German books and the Spanish books first. Putting these $5+7=12$ books creates $12+1=13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that all the French books are next each other, we "glue" them together and put them in one of these spaces. Now, the French books can be permuted in $8!$ ways and the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is

$$(13)8!12! = 251073478656000.$$

❹ Align the German books and the Spanish books first. Putting these $5+7=12$ books creates $12+1=13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that no two French books are next to each other, we put them into these spaces. The first French book can be put into any of 13 spaces, the second into any of 12, etc., the eighth French book can be put into any 6 spaces. Now, the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is

$$(13)(12)(11)(10)(9)(8)(7)(6)12! = 24856274386944000.$$

235 Example In how many ways can 8 people be seated in a row if

❶ there are no constraints as to their seating arrangement?

❷ persons X and Y must sit next to one another?

❸ there are 4 women and 4 men and no 2 men or 2 women can sit next to each other?

❹ there are 4 married couples and each couple must sit together?

❺ there are 4 men and they must sit next to each other?

Solution:

❶ This is $8!$.

❷ Permute XY in $2!$ and put them in any of the 7 spaces created by the remaining 6 people. Permute the remaining 6 people. This is $2!\cdot 7\cdot 6!$.

❸ In this case, we alternate between sexes. Either we start with a man or a woman (giving 2 ways), and then we permute the men and the women. This is $2 \cdot 4!4!$.

❹ Glue the couples into 4 separate blocks. Permute the blocks in $4!$ ways. Then permute each of the 4 blocks in $2!$. This is $4!(2!)^4$.

❺ Sit the women first, creating 5 spaces in between. Glue the men together and put them in any of the 5 spaces. Permute the men in $4!$ ways and the women in $4!$. This is $5 \cdot 4!4!$.

Homework

236 Problem How many changes can be rung with a peal of five bells?

Answer: 120

237 Problem A bookshelf contains 3 Russian novels, 4 German novels, and 5 Spanish novels. In how many ways may we align them if

❶ there are no constraints as to grouping?

❷ all the Spanish novels must be together?

❸ no two Spanish novels are next to one another?

Answer: 479001600; 4838400; 33868800

238 Problem How many permutations of the word **IMPURE** are there? How many permutations start with **P** and end in **U**? How many permutations are there if the **P** and the **U** must always be together in the order **PU**? How many permutations are there in which no two vowels (**I, U, E**) are adjacent?

Answer: 720; 24; 120; 144

239 Problem How many arrangements can be made of out of the letters of the word **DRAUGHT**, the vowels never separated?

Answer: 1440

240 Problem (AIME 1991) Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will $20!$ be the resulting product?

Answer: 128

241 Problem (AMC12 2001) A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

Answer: 81729648000

242 Problem How many trailing 0's are there when $1000!$ is multiplied out?

Answer: 249

2.5 Permutations with Repetitions

We now consider permutations with repeated objects.

243 Example In how many ways may the letters of the word

$$MASSACHUSETTS$$

be permuted?

Solution: We put subscripts on the repeats forming

$$MA_1S_1S_2A_2CHUS_3ET_1T_2S_4.$$

There are now 13 distinguishable objects, which can be permuted in $13!$ different ways by Theorem 232. For each of these $13!$ permutations, A_1A_2 can be permuted in $2!$, $S_1S_2S_3S_4$ can be permuted in $4!$, and T_1T_2 can be permuted in $2!$. Thus the over count $13!$ is corrected by the total actual count

$$\frac{13!}{2!4!2!} = 64864800.$$

A reasoning analogous to the one of example 243, we may prove

244 Theorem Let there be k types of objects: n_1 of type 1; n_2 of type 2; etc. Then the number of ways in which these $n_1 + n_2 + \cdots + n_k$ objects can be rearranged is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}.$$

245 Example In how many ways may we permute the letters of the word $MASSACHUSETTS$ in such a way that $MASS$ is always together, in this order?

Solution: The particle $MASS$ can be considered as one block and the 9 letters A, C, H, U, S, E, T, T, S. In A, C, H, U, S, E, T, T, S there are four S's and two T's and so the total number of permutations sought is

$$\frac{10!}{2!2!} = 907200.$$

246 Example In this problem you will determine how many different signals, each consisting of 10 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, 2 blue flags, and 1 orange flag, if flags of the same colour are identical.

❶ How many are there if there are no constraints on the order?

❷ How many are there if the orange flag must always be first?

❸ How many are there if there must be a white flag at the beginning and another white flag at the end?

Solution:

❶ This is
$$\frac{10!}{4!3!2!}$$

❷ This is
$$\frac{9!}{4!3!2!}$$

❸ This is
$$\frac{8!}{2!3!2!}$$

247 Example In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1+7+1$ is to be regarded different from $7+1+1$.

Solution: We first look for answers with
$$a+b+c=9, 1 \leq a \leq b \leq c \leq 7$$

and we find the permutations of each triplet. We have

(a, b, c)	Number of permutations
$(1, 1, 7)$	$\frac{3!}{2!} = 3$
$(1, 2, 6)$	$3! = 6$
$(1, 3, 5)$	$3! = 6$
$(1, 4, 4)$	$\frac{3!}{2!} = 3$
$(2, 2, 5)$	$\frac{3!}{2!} = 3$
$(2, 3, 4)$	$3! = 6$
$(3, 3, 3)$	$\frac{3!}{3!} = 1$

Thus the number desired is

$$3 + 6 + 6 + 3 + 3 + 6 + 1 = 28.$$

248 Example In how many ways can the letters of the word **MURMUR** be arranged without letting two letters which are alike come together?

Solution: If we started with, say , **MU** then the **R** could be arranged as follows:

M	**U**	**R**		**R**	

,

M	**U**	**R**			**R**

,

M	**U**		**R**		**R**

.

In the first case there are $2! = 2$ of putting the remaining **M** and **U**, in the second there are $2! = 2$ and in the third there is only $1!$. Thus starting the word with **MU** gives $2+2+1=5$ possible arrangements. In the general case, we can choose the first letter of the word in 3 ways, and the second in 2 ways. Thus the number of ways sought is $3\cdot 2\cdot 5=30$.

249 Example In how many ways can the letters of the word **AFFECTION** be arranged, keeping the vowels in their natural order and not letting the two **F**'s come together?

Solution: There are $\frac{9!}{2!}$ ways of permuting the letters of **AFFECTION**. The 4 vowels can be permuted in $4!$ ways, and in only one of these will they be in their natural order. Thus there are $\frac{9!}{2!4!}$ ways of permuting the letters of **AFFECTION** in which their vowels keep their natural order.

Now, put the 7 letters of **AFFECTION** which are not the two **F**'s. This creates 8 spaces in between them where we put the two **F**'s. This means that there are $8\cdot 7!$ permutations of **AFFECTION** that keep the two **F**'s together. Hence there are $\frac{8\cdot 7!}{4!}$ permutations of **AFFECTION** where the vowels occur in their natural order.

In conclusion, the number of permutations sought is

$$\frac{9!}{2!4!}-\frac{8\cdot 7!}{4!}=\frac{8!}{4!}\left(\frac{9}{2}-1\right)=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!}\cdot\frac{7}{2}=5880$$

250 Example How many arrangements of five letters can be made of the letters of the word **PALLMALL**?

Solution: We consider the following cases:

❶ there are four **L**'s and a different letter. The different letter can be chosen in 3 ways, so there are $\frac{3\cdot 5!}{4!}=15$ permutations in this case.

❷ there are three **L**'s and two **A**'s. There are $\frac{5!}{3!2!}=10$ permutations in this case.

❸ there are three **L**'s and two different letters. The different letters can be chosen in 3 ways (either **P** and **A**; or **P** and **M**; or **A** and **M**), so there are $\frac{3\cdot 5!}{3!} = 60$ permutations in this case.

❹ there are two **L**'s, two **A**'s and a different letter from these two. The different letter can be chosen in 2 ways. There are $\frac{2\cdot 5!}{2!2!} = 60$ permutations in this case.

❺ there are two **L**'s and three different letters. The different letters can be chosen in 1 way. There are $\frac{1\cdot 5!}{2!} = 60$ permutations in this case.

❻ there is one **L**. This forces having two **A**'s and two other different letters. The different letters can be chosen in 1 way. There are $\frac{1\cdot 5!}{2!} = 60$ permutations in this case.

The total number of permutations is thus seen to be

$$15 + 10 + 60 + 60 + 60 + 60 = 265.$$

Homework

251 Problem In how many ways may one permute the letters of the word **MEPHISTOPHELES**?

Answer: 1816214400

252 Problem How many arrangements of four letters can be made out of the letters of **KAFFEEKANNE** without letting the three **E**'s come together?

Answer: 548

253 Problem How many numbers can be formed with the digits

$$1, 2, 3, 4, 3, 2, 1$$

so that the odd digits occupy the odd places?

Answer: 18

254 **Problem** In how many ways may we write the number 10 as the sum of three positive integer summands? Here order counts, so, for example, $1 + 8 + 1$ is to be regarded different from $8 + 1 + 1$.

Answer: 36

255 **Problem** Three distinguishable dice are thrown. In how many ways can they land and give a sum of 9?

Answer: 25

256 **Problem** In how many ways can 15 different recruits be divided into three equal groups? In how many ways can they be drafted into three different regiments?

Answer: 126126; 756756

2.6 Combinations without Repetitions

257 Definition Let n, k be non-negative integers with $0 \leq k \leq n$. The symbol $\binom{n}{k}$ (read "n *choose* k") is defined and denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

Observe that in the last fraction, there are k factors in both the numerator and denominator. Also, observe that

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

258 Example

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20,$$

$$\binom{11}{2} = \frac{11 \cdot 10}{1 \cdot 2} = 55,$$

$$\binom{12}{7} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 792,$$

$$\binom{110}{109} = 110,$$

$$\binom{110}{0} = 1.$$

Since $n - (n-k) = k$, we have for integer n, k, $0 \leq k \leq n$, the symmetry identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

This can be interpreted as follows: if there are n different tickets in a hat, choosing k of them out of the hat is the same as choosing $n-k$ of them to remain in the hat.

259 Example

$$\binom{11}{9} = \binom{11}{2} = 55,$$

$$\binom{12}{5} = \binom{12}{7} = 792.$$

260 Definition Let there be n distinguishable objects. A k-*combination* is a selection of k, $(0 \leq k \leq n)$ objects from the n made without regards to order.

261 Example The 2-combinations from the list $\{X, Y, Z, W\}$ are

$$\{X, Y\}, \{X, Z\}, \{X, W\}, \{Y, Z\}, \{Y, W\}, \{W, Z\}.$$

262 Example The 3-combinations from the list $\{X, Y, Z, W\}$ are

$$\{X, Y, Z\}, \{X, Y, W\}, \{X, Z, W\}, \{Y, W, Z\}.$$

263 Theorem Let there be n distinguishable objects, and let k, $0 \leq k \leq n$. Then the numbers of k-combinations of these n objects is $\binom{n}{k}$.

Proof Pick any of the k objects. They can be ordered in $n(n-1)(n-2)\cdots(n-k+1)$, since there are n ways of choosing the *first*, $n-1$ ways of choosing the *second*, etc. This particular choice of k objects can be permuted in $k!$ ways. Hence the total number of k-combinations is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

264 Example From a group of 10 people, we may choose a committee of 4 in $\binom{10}{4} = 210$ ways.

265 Example The number of subsets of the set $\{a, b, c, d, e\}$ containing 3 elements is $\binom{5}{3} = 10$.

266 Example The number of subsets of the set $\{a, b, c, d, e\}$ containing an odd number of elements is

$$\binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 5 + 10 + 1 = 16.$$

In a set of n elements there are 2^n subsets by virtue of Theorem 186. Half of these subsets, that is, $\frac{2^n}{2} = 2^{n-1}$ contain an even number of elements and the other half contain an odd number of elements. Thus

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n},$$

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

and

$$2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

267 Example From a group of 20 students, in how many ways may a professor choose at least one in order to work on a project?

Solution: The required number is

$$\binom{20}{1}+\binom{20}{2}+\cdots+\binom{20}{20}=2^{20}-\binom{20}{0}=1048576-1=1048575.$$

268 Example From a group of 20 students, in how many ways may a professor choose an even number number of them, but at least four in order to work on a project?

Solution: The required number is

$$\binom{20}{4}+\binom{20}{6}+\cdots+\binom{20}{20}=2^{19}-\binom{20}{0}-\binom{20}{2}=524288-1-190=524097.$$

269 Example From a group of 12 people—7 of which are men and 5 women—we may choose a committee of 4 with 1 man and 3 women in

$$\binom{7}{1}\binom{5}{3}=(7)(10)=70$$

ways.

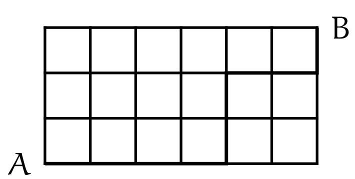


Figure 2.8: Example 270.

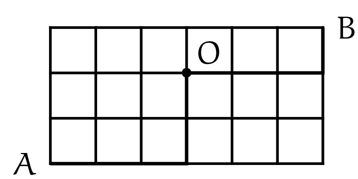


Figure 2.9: Example 271.

270 Example To count the number of shortest routes from A to B in figure 2.8 observe that any shortest path must consist of 6 horizontal moves and 3 vertical ones for a total of $6+3=9$ moves. Of these 9 moves once we choose the 6 horizontal ones the 3 vertical ones are determined. Thus there are $\binom{9}{6} = 84$ paths.

271 Example To count the number of shortest routes from A to B in figure 2.9 that pass through point O we count the number of paths from A to O (of which there are $\binom{5}{3} = 20$) and the number of paths from O to B (of which there are $\binom{4}{3} = 4$). Thus the desired number of paths is $\binom{5}{3}\binom{4}{3} = (20)(4) = 80$.

272 Example How many committees of seven with a given chairman can be selected from twenty people?

Solution: We can choose the seven people in $\binom{20}{7}$ ways. Of the seven, the chairman can be chosen in seven ways. The answer is thus

$$7\binom{20}{7} = 542640.$$

Aliter: Choose the chairman first. This can be done in twenty ways. Out of the nineteen remaining people, we just have to choose six, this can be done in $\binom{19}{6}$ ways. The total number of ways is hence $20\binom{19}{6} = 542640$.

273 Example How many committees of seven with a given chairman and a given secretary can be selected from twenty people? Assume the chairman and the secretary are different persons.

Solution: We can choose the seven people in $\binom{20}{7}$ ways. Of these seven people chosen, we can choose the chairman in seven ways and the secretary in six ways. The answer is thus $7 \cdot 6\binom{20}{7} = 3255840$.

Aliter: If one chooses the chairman first, then the secretary and finally the remaining five people of the committee, this can be done in $20 \cdot 19 \cdot \binom{18}{5} = 3255840$ ways.

274 Example In a group of 2 camels, 3 goats, and 10 sheep in how many ways may one choose 6 animals if

❶ there are no constraints in species?

❷ the two camels must be included?

❸ the two camels must be excluded?

❹ there must be at least 3 sheep?

❺ there must be at most 2 sheep?

❻ Joe Camel, Billy Goat and Samuel Sheep hate each other and they will not work in the same group. How many compatible committees are there?

Solution:

❶ $\binom{15}{6} = 5005$

❷ $\binom{13}{4} = 715$

❸ $\binom{13}{6} = 1716$

❹ $\binom{10}{3}\binom{5}{3} + \binom{10}{4}\binom{5}{2} + \binom{10}{5}\binom{5}{1} + \binom{10}{6}\binom{5}{0} = 4770$

❺ $\binom{10}{2}\binom{5}{4} + \binom{10}{1}\binom{5}{5} = 235$

❻ A compatible group will either exclude all these three animals—which can be done in $\binom{12}{6} = 924$ ways—or include exactly one of them—which can be done in $\binom{3}{1}\binom{12}{5} = 2376$. Thus the total is $2376 + 924 = 3300$.

275 Example The number of lists of 3 elements taken from the set $\{1, 2, 3, 4, 5, 6\}$ that list the elements in increasing order is $\binom{6}{3} = 20$.

276 Example (AHSME 1990) How many of the numbers

$$100, 101, \ldots, 999,$$

have three different digits in increasing order or in decreasing order?

Solution: For a string of three-digit numbers to be decreasing, the digits must come from $\{0, 1, \ldots, 9\}$ and so there are $\binom{10}{3} = 120$ three-digit numbers with all its digits in decreasing order. If the string of three-digit numbers is increasing, the digits have to come from $\{1, 2, \ldots, 9\}$, thus there are $\binom{9}{3} = 84$ three-digit numbers with all the digits increasing. The total asked is hence $120 + 84 = 204$.

277 Example There are twenty students in a class. In how many ways can the twenty students take five different tests if four of the students are to take each test?

Solution: We can choose the four students who are going to take the first test in $\binom{20}{4}$ ways. From the remaining ones, we can choose students in $\binom{16}{4}$ ways to take the second test. The third test can be taken in $\binom{12}{4}$ ways. The fourth in $\binom{8}{4}$ ways and the fifth in $\binom{4}{4}$

ways. The total number is thus

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}.$$

278 Example How many times is the digit 3 listed in the numbers 1 to 1000?

Solution: We count those numbers that have exactly once, twice and three times. There is only one number that has it thrice (namely 333). Suppose the number xyz is to have the digit 3 exactly twice. We can choose these two positions in $\binom{3}{2}$ ways. The third position can be filled with any of the remaining nine digits (the digit 3 has already been used). Thus there are $9\binom{3}{2}$ numbers that the digit 3 exactly twice. Similarly, there are $9^2\binom{3}{2}$ numbers that have 3 exactly once. The total required is hence $3 \cdot 1 + 2 \cdot 9 \cdot \binom{3}{2} + 9^2\binom{3}{1} = 300$.

279 Example Consider the set of 5-digit positive integers written in decimal notation.

1. How many are there?
2. How many do not have a 9 in their decimal representation?
3. How many have at least one 9 in their decimal representation?
4. How many have exactly one 9?
5. How many have exactly two 9's?
6. How many have exactly three 9's?
7. How many have exactly four 9's?
8. How many have exactly five 9's?
9. How many have neither an 8 nor a 9 in their decimal representation?

10. How many have neither a 7, nor an 8, nor a 9 in their decimal representation?

11. How many have either a 7, an 8, or a 9 in their decimal representation?

Solution:

1. There are 9 possible choices for the first digit and 10 possible choices for the remaining digits. The number of choices is thus $9 \cdot 10^4 = 90000$.

2. There are 8 possible choices for the first digit and 9 possible choices for the remaining digits. The number of choices is thus $8 \cdot 9^4 = 52488$.

3. The difference $90000 - 52488 = 37512$.

4. We condition on the first digit. If the first digit is a 9 then the other four remaining digits must be different from 9, giving $9^4 = 6561$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{1} = 4$ ways of choosing were the 9 will be, and we have 9^3 ways of filling the 3 remaining spots. Thus in this case there are $8 \cdot 4 \cdot 9^3 = 23328$ such numbers. In total there are $6561 + 23328 = 29889$ five-digit positive integers with exactly one 9 in their decimal representation.

5. We condition on the first digit. If the first digit is a 9 then one of the remaining four must be a 9, and the choice of place can be accomplished in $\binom{4}{1} = 4$ ways. The other three remaining digits must be different from 9, giving $4 \cdot 9^3 = 2916$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{2} = 6$ ways of choosing were the two 9's will be, and we have 9^2 ways of filling the two remaining

spots. Thus in this case there are $8\cdot 6\cdot 9^2 = 3888$ such numbers. Altogether there are $2916 + 3888 = 6804$ five-digit positive integers with exactly two 9's in their decimal representation.

6. Again we condition on the first digit. If the first digit is a 9 then two of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{2} = 6$ ways. The other two remaining digits must be different from 9, giving $6 \cdot 9^2 = 486$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{3} = 4$ ways of choosing were the three 9's will be, and we have 9 ways of filling the remaining spot. Thus in this case there are $8 \cdot 4 \cdot 9 = 288$ such numbers. Altogether there are $486 + 288 = 774$ five-digit positive integers with exactly three 9's in their decimal representation.

7. If the first digit is a 9 then three of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{3} = 4$ ways. The other remaining digit must be different from 9, giving $4 \cdot 9 = 36$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{4} = 4$ ways of choosing were the four 9's will be, thus filling all the spots. Thus in this case there are $8 \cdot 1 = 8$ such numbers. Altogether there are $36 + 8 = 44$ five-digit positive integers with exactly three 9's in their decimal representation.

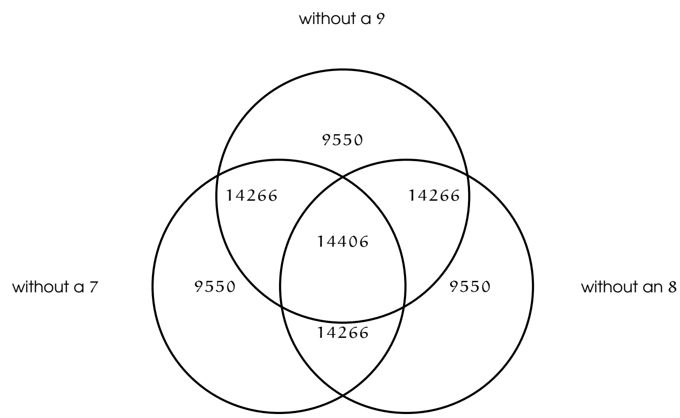
8. There is obviously only 1 such positive integer.

REMARK: Observe that $37512 = 29889 + 6804 + 774 + 44 + 1$.

9. We have 7 choices for the first digit and 8 choices for the remaining 4 digits, giving $7 \cdot 8^4 = 28672$ such integers.

10. We have 6 choices for the first digit and 7 choices for the remaining 4 digits, giving $6 \cdot 7^4 = 14406$ such integers.

11. We use inclusion-exclusion.



The numbers inside the circles add up to 85854. Thus the desired number is $90000 - 85854 = 4146$.

280 Example There are T books on Theology, L books on Law and W books on Witchcraft on Dr. Faustus' shelf. In how many ways may one order the books

❶ there are no constraints in their order?

❷ all books of a subject must be together?

❸ no two books on Witchcraft are juxtaposed?

❹ all the books on Witchcraft must be together?

Solution:

❶ $(T + L + W)!$

❷ $3!T!L!W! = 6T!L!W!$

❸ $\binom{T+L+1}{W}(T + L)!W!$

❹ $\binom{T+L+1}{1}(T + L)!W!$

281 Example In how many ways can a deck of playing cards be arranged if no two hearts are adjacent?

Solution: We align the thirty-nine cards which are not hearts first. There are thirty-eight spaces between them and one at the beginning and one at the end making a total of forty spaces where the hearts can go. Thus there are $\binom{40}{13}$ ways of choosing the *places* where the hearts can go. Now, since we are interested in arrangements, there are $39!$ different configurations of the non-hearts and $13!$ different configurations of the hearts. The total number of arrangements is thus $\binom{40}{13}39!13!$.

282 Example Given a positive integer n, find the number of quadruples $(a, b, c, d,)$ such that $0 \leq a \leq b \leq c \leq d \leq n$.

Solution: The equality signs cause us trouble, since allowing them would entail allowing repetitions in our choices. To overcome that we establish a one-to-one correspondence between the vectors $(a, b, c, d), 0 \leq a \leq b \leq c \leq d \leq n$ and the vectors $(a', b', c', d'), 0 \leq a' < b' < c' < d' \leq n+3$. Let $(a', b', c', d') = (a, b+1, c+2, d+3)$. Now we just have to pick four different numbers from the set $\{0, 1, 2, 3, \ldots, n, n+1, n+2, n+3\}$. This can be done in $\binom{n+4}{4}$ ways.

Homework

283 Problem Verify the following.

❶ $\binom{20}{3} = 1140$

❷ $\binom{12}{4}\binom{12}{6} = 457380$

❸ $\frac{\binom{n}{1}}{\binom{n}{n-1}} = 1$

❹ $\binom{n}{2} = \frac{n(n-1)}{2}$

❺ $\binom{6}{1} + \binom{6}{3} + \binom{6}{6} = 2^5.$

❻ $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} = 2^6 - \binom{7}{6}$

284 Problem A publisher proposes to issue a set of dictionaries to translate from any one language to any other. If he confines his system to ten languages, how many dictionaries must be published?

Answer: $\binom{10}{2} = 45$

285 Problem N friends meet and shake hands with one another. How many handshakes?

$\binom{N}{2}$

286 Problem How many 4-letter words can be made by taking 4 letters of the word **RETICULA** and permuting them?

Answer: $\binom{8}{4}4! = 1680$

287 Problem (AHSME 1989) Mr. and Mrs. Zeta want to name baby Zeta so that its monogram (first, middle and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

Answer: $\binom{25}{2} = 300$

288 Problem (AHSME 1994) Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

Answer: $10 \times 3! = 60$

289 Problem There are E (different) English novels, F (different) French novels, S (different) Spanish novels, and I (different) Italian novels on a shelf. How many different permutations are there if

❶ if there are no restrictions?

❷ if all books of the same language must be together?

❸ if all the Spanish novels must be together?

❹ if no two Spanish novels are adjacent?

❺ if all the Spanish novels must be together, and all the English novels must be together, but no Spanish novel is next to an English novel?

Answer: $(E + F + S + I)!$; $4! \cdot E!F!S!I!$; $\binom{E+F+I+1}{1} S!(E + F + I)!$; $\binom{E+F+I+1}{S} S!(E+F+I)!$; $2!\binom{F+I+1}{2} S!E!(F+I)!$

290 Problem How many permutations of the word

CHICHICUILOTE

are there

❶ if there are no restrictions?

❷ if the word must start in an **I** and end also in an **I**?

❸ if the word must start in an **I** and end in a **C**?

❹ if the two **H**'s are adjacent?

❺ if the two **H**'s are not adjacent?

❻ if the particle **LOTE** must appear, with the letters in this order?

Answer: $\frac{13!}{2!3!3!} = 86486400$; $\frac{11!}{2!3!} = 3326400$; $\frac{11!}{2!2!2!} = 4989600$; $\binom{12}{1}\frac{11!}{3!3!} = 13305600$; $\binom{12}{2}\frac{11!}{3!3!} = 73180800$; $\binom{10}{1}\frac{9!}{3!3!2!} = 50400$

291 Problem There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if

❶ there are no constraints on the sex of the committee members?

❷ there must be exactly T women?

❸ A committee must always include George and Barbara?

❹ A committee must always exclude George and Barbara?

Assume George and Barbara form part of the original set of people.

Answer: $\binom{M+W}{C}$; $\binom{M}{C-T}\binom{W}{T}$; $\binom{M+W-2}{C-2}$; $\binom{M+W-2}{C}$

292 Problem There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if George and Barbara are feuding and will not work together in a committee? Assume George and Barbara form part of the original set of people.

Answer:

$$\binom{M+W}{C} - \binom{M+W-2}{C-2} = 2\binom{M+W-2}{C-1} + \binom{M+W-2}{C}.$$

293 Problem Out of 30 consecutive integers, in how many ways can three be selected so that their sum be even?

Answer: 2030

294 Problem In how many ways may we choose three distinct integers from $\{1, 2, \ldots, 100\}$ so that one of them is the average of the other two?

Answer: $2\binom{50}{2}$

295 Problem How many vectors $(a_1, a_2, \ldots, a_k)$ with integral

$$a_i \in \{1, 2, \ldots, n\}$$

are there satisfying

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n?$$

Answer: $\binom{n+k-1}{k}$

296 Problem A box contains 4 red, 5 white, 6 blue, and 7 magenta balls. In how many of all possible samples of size 5, chosen without replacement, will every colour be represented?

Answer: 7560.

297 Problem In how many ways can eight students be divided into four indistinguishable teams of two each?

Answer: $\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}$.

298 Problem How many ways can three boys share fifteen different sized pears if the youngest gets seven pears and the other two boys get four each?

Answer: $\binom{15}{7}\binom{8}{4}$.

299 Problem Among the integers 1 to 10^{10}, which are there more of: those in which the digit 1 occurs or those in which it does not occur?

Answer: There are 6513215600 of former and 3486784400 of the latter.

300 Problem Four writers must write a book containing seventeen chapters. The first and third writers must each write five chapters, the second must write four chapters, and the fourth must write three chapters. How many ways can the book be divided between the authors? What if the first and third had to write ten chapters combined, but it did not matter which of them wrote how many (i.e. the first could write ten and the third none, the first could write none and the third one, etc.)?

Answer: $\binom{17}{5}\binom{12}{5}\binom{7}{4}\binom{3}{3}$; $\binom{17}{3}\binom{14}{4}2^{10}$.

301 Problem In how many ways can a woman choose three lovers or more from seven eligible suitors?

Answer: $\sum_{k=3}^{7}\binom{7}{k} = 99$

302 Problem (AIME 1988) One commercially available ten-button lock may be opened by depressing—in any order—the correct five buttons. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

Answer: $2^{10} - 1 - 1 - \binom{10}{5} = 1024 - 2 - 252 = 770$

303 Problem From a set of $n \geq 3$ points on the plane, no three collinear, ❶ how many straight lines are determined?

❷ how many straight lines pass through a particular point?

❸ how many triangles are determined?

❹ how many triangles have a particular point as a vertex?

Answer: $\binom{n}{2}$; $n-1$; $\binom{n}{3}$; $\binom{n-1}{2}$

304 Problem In how many ways can you pack twelve books into four parcels if one parcel has one book, another has five books, and another has two books, and another has four books?

Answer: $\binom{12}{1}\binom{11}{5}\binom{6}{2}\binom{4}{4}$

305 Problem In how many ways can a person invite three of his six friends to lunch every day for twenty days if he has the option of inviting the same or different friends from previous days?

Answer: $\binom{6}{3}^{20} = 104857600000000000000000000$

306 Problem A committee is to be chosen from a set of nine women and five men. How many ways are there to form the committee if the committee has three men and three women?

Answer: $\binom{9}{3}\binom{5}{3} = 840$

307 Problem At a dance there are b boys and g girls. In how many ways can they form c couples consisting of different sexes?

Answer: $\binom{b}{c}\binom{g}{c}c!$

308 Problem From three Russians, four Americans, and two Spaniards, how many selections of people can be made, taking at least one of each kind?

Answer: $(2^3 - 1)(2^4 - 1)(2^2 - 1) = 315$

309 Problem The positive integer r satisfies

$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}.$$

Find r.

310 Problem If $11\binom{28}{2r} = 225\binom{24}{2r-4}$, find r.

311 Problem Compute the number of ten-digit numbers which contain only the digits $1, 2$, and 3 with the digit 2 appearing in each number exactly twice.

Answer: $\binom{10}{2}2^8$

312 Problem In each of the 6-digit numbers

$$333333, 225522, 118818, 707099,$$

each digit in the number appears at least twice. Find the number of such 6-digit natural numbers.

Answer: 11754

313 Problem In each of the 7-digit numbers

$$1001011, 5550000, 3838383, 7777777,$$

each digit in the number appears at least thrice. Find the number of such 7-digit natural numbers.

Answer: 2844

314 Problem (AIME 1983) The numbers 1447, 1005 and 1231 have something in common: each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

Answer: 432

315 Problem If there are fifteen players on a baseball team, how many ways can the coach choose nine players for the starting lineup if it does not matter which position the players play (i.e., no distinction is made between player A playing shortstop, left field, or any other positions as long as he is on the field)? How many ways are there if it does matter which position the players play?

Answer: $\binom{15}{9}; 15!/6!$

316 Problem (AHSME 1989) A child has a set of 96 distinct blocks. Each block is one of two materials (*plastic, wood*), three sizes (*small, medium, large*), four colours (*blue, green, red, yellow*), and four shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the "*plastic medium red circle*" in exactly two ways? (The "*wood medium red square*" is such a block.)

Answer: 29.

317 Problem (AHSME 1989) Suppose that k boys and $n-k$ girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row

$$\text{GBBGGGBGBGGGBGBGGBGG},$$

with $k = 7, n = 20$ we have $S = 12$. Shew that the average value of S is $\frac{2k(n-k)}{n}$.

318 Problem There are four different kinds of sweets at a sweets store. I want to buy up to four sweets (I'm not sure if I want none, one, two, three, or four sweets) and I refuse to buy more than one of any kind of sweet. How many ways can I do this?

Answer: 2^4

319 Problem Suppose five people are in a lift. There are eight floors that the lift stops at. How many distinct ways can the people exit the lift if either one or zero people exit at each stop?

Answer: $\binom{8}{5}5!$

320 Problem If the natural numbers from 1 to 222222222 are written down in succession, how many 0's are written?

Answer: 175308642

321 Problem In how many ways can we distribute k identical balls into n different boxes so that each box contains at most one ball and no two consecutive boxes are empty?

Hint: There are k occupied boxes and $n-k$ empty boxes. Align the balls first! Answer: $\binom{k+1}{n-k}$.

322 Problem In a row of n seats in the doctor's waiting-room k patients sit down in a particular order from left to right. They sit so that no two of them are in adjacent seats. In how many ways could a suitable set of k seats be chosen?
Hint: There are $n-k$ empty seats. Sit the people in between those seats. Answer: $\binom{n-k+1}{k}$.

2.7 Combinations with Repetitions

323 Theorem (De Moivre) Let n be a positive integer. The number of positive integer solutions to

$$x_1 + x_2 + \cdots + x_r = n$$

is

$$\binom{n-1}{r-1}.$$

Proof Write n as

$$n = 1 + 1 + \cdots + 1 + 1,$$

where there are n 1s and $n-1$ +s. To decompose n in r summands we only need to choose $r-1$ pluses from the $n-1$, which proves the theorem. ❑

324 Example In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1+7+1$ is to be regarded different from $7+1+1$.

Solution: Notice that this is example 247. We are seeking integral solutions to

$$a + b + c = 9, \quad a > 0, b > 0, c > 0.$$

By Theorem 323 this is

$$\binom{9-1}{3-1} = \binom{8}{2} = 28.$$

325 Example In how many ways can 100 be written as the sum of four positive integer summands?

Solution: We want the number of positive integer solutions to

$$a + b + c + d = 100,$$

which by Theorem 323 is

$$\binom{99}{3} = 156849.$$

326 Corollary Let n be a positive integer. The number of non-negative integer solutions to

$$y_1 + y_2 + \cdots + y_r = n$$

is

$$\binom{n+r-1}{r-1}.$$

Proof Put $x_r - 1 = y_r$. Then $x_r \geq 1$. The equation

$$x_1 - 1 + x_2 - 1 + \cdots + x_r - 1 = n$$

is equivalent to

$$x_1 + x_2 + \cdots + x_r = n + r,$$

which from Theorem 323, has

$$\binom{n+r-1}{r-1}$$

solutions. ❑

327 Example Find the number of quadruples (a, b, c, d) of integers satisfying

$$a + b + c + d = 100, \ a \geq 30, b > 21, c \geq 1, d \geq 1.$$

Solution: Put $a' + 29 = a, b' + 20 = b$. Then we want the number of positive integer solutions to

$$a' + 29 + b' + 21 + c + d = 100,$$

or

$$a' + b' + c + d = 50.$$

By Theorem 323 this number is

$$\binom{49}{3} = 18424.$$

328 Example There are five people in a lift of a building having eight floors. In how many ways can they choose their floor for exiting the lift?

Solution: Let x_i be the number of people that floor i receives. We are looking for non-negative solutions of the equation

$$x_1 + x_2 + \cdots + x_8 = 5.$$

Putting $y_i = x_i + 1$, then

$$\begin{aligned} x_1 + x_2 + \cdots + x_8 = 5 \quad & \implies \quad (y_1 - 1) + (y_2 - 1) + \cdots + (y_8 - 1) = 5 \\ & \implies \quad y_1 + y_2 + \cdots + y_8 = 13, \end{aligned}$$

whence the number sought is the number of positive solutions to

$$y_1 + y_2 + \cdots + y_8 = 13$$

which is $\binom{12}{7} = 792$.

329 Example Find the number of quadruples (a, b, c, d) of non-negative integers which satisfy the inequality

$$a + b + c + d \leq 2001.$$

Solution: The number of non-negative solutions to

$$a + b + c + d \leq 2001$$

equals the number of solutions to

$$a + b + c + d + f = 2001$$

where f is a non-negative integer. This number is the same as the number of positive integer solutions to

$$a_1 - 1 + b_1 - 1 + c_1 - 1 + d_1 - 1 + f_1 - 1 = 2001,$$

which is easily seen to be $\binom{2005}{4}$.

Homework

330 Problem How many positive integral solutions are there to

$$a + b + c = 10?$$

Answer: 36

331 Problem Three fair dice, one red, one white, and one blue are thrown. In how many ways can they land so that their sum be 10 ?

Answer: 27

332 Problem Adena has twenty indistinguishable pieces of sweet-meats that she wants to divide amongst her five stepchildren. How many ways can she divide the sweet-meats so that each stepchild gets at least two pieces of sweet-meats?

Answer: $\binom{14}{4}$

333 Problem How many integral solutions are there to the equation

$$x_1 + x_2 + \cdots + x_{100} = n$$

subject to the constraints

$$x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, \ldots, x_{99} \geq 99, x_{100} \geq 100?$$

334 Problem (AIME 1998) Find the number of ordered quadruplets (a, b, c, d) of positive odd integers satisfying $a + b + c + d = 98$.

Answer: $\binom{50}{3} = 19600$

2.8 Binomial Theorem

335 Example Expand $(x+1)(x-2)(x+3)(x-4)$.

Solution: To obtain the x^4 term we take an x from each of the parentheses, to obtain the x^3 we take one constant and three x's, etc. Thus

$$\begin{aligned}(x+1)(x-2)(x+3)(x-4) &= \\ &= x^4+(1-2+3-4)x^3 \\ &\qquad +(-2+3-4-6+8-12)x^2 \\ &\qquad +(-6+8-12+24)x+24 \\ &= x^4-2x^3-13x^2+14x+24.\end{aligned}$$

We recall that the symbol

$$\binom{n}{k}=\frac{n!}{(n-k)!k!}, \quad n,k\in\mathbb{N}, 0\le k\le n,$$

counts the number of ways of selecting k different objects from n different objects.

336 Example Prove *Pascal's Identity*:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k},$$

for integers $1\le k\le n$.

Solution: We have

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\frac{n}{(n-k)k}\\ &= \frac{n!}{(n-k)!k!}.\\ &= \binom{n}{k}\end{aligned}$$

Using Pascal's Identity we obtain *Pascal's Triangle.*

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

When the binomial symbols are evaluated, the triangle then looks like this.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

The *Binomial Theorem* states that for $n \in \mathbb{Z}, n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

As a way of proving this, we observe that expanding

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

consists of adding up all the terms obtained from multiplying either an x or a y from the first set of parentheses times either an x or a y from the second set of parentheses etc. To get x^k, x must be chosen from exactly k of the sets of parentheses. Thus the number of x^k terms is $\binom{n}{k}$. It follows that

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}xy^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.\end{aligned} \tag{2.1}$$

By setting $x = y = 1$ in 2.1 we obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n},$$

337 Example Expand $(2-x)^5$.

Solution: By the Binomial Theorem

$$(2-x)^5 = \sum_{k=0}^{5} 2^{5-k}(-x)^k \binom{5}{k} = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5.$$

338 Example Prove that $t^3 = (t-2)^3 + 6(t-2)^2 + 12(t-2) + 8$.

Solution: By the Binomial Theorem, putting $x = t-2, y = 2$

$$\begin{aligned} t^3 &= (t-2+2)^3 \\ &= \binom{3}{0}(t-2)^3 + \binom{3}{1}(t-2)^2(2) + \binom{3}{2}(t-2)(2)^2 + 2^3 \\ &= (t-2)^3 + 6(t-2)^2 + 12(t-2) + 8, \end{aligned}$$

as required.

339 Example Simplify

$$\sum_{k=1}^{10} 2^k \binom{11}{k}.$$

Solution: By the Binomial Theorem, the complete sum $\sum_{k=0}^{11} \binom{11}{k} 2^k = 3^{11}$. The required sum lacks the zeroth term, $\binom{11}{0} 2^0 = 1$, and the eleventh term, $\binom{11}{11}2^{11}$ from this complete sum. The required sum is thus $3^{11} - 2^{11} - 1$.

340 Example Find the coefficient of x^{12} in the expansion of

$$(x^2 + 2x)^{10}.$$

Solution: We have

$$(x^2 + 2x)^{10} = \sum_{k=0}^{10} \binom{10}{k} (x^2)^k (2x)^{10-k} = \sum_{k=0}^{10} \binom{10}{k} 2^{10-k} x^{k+10}.$$

To obtain x^{12} we need $k = 2$. Hence the coefficient sought is

$$\binom{10}{2} 2^8 = 11520.$$

Here is another proof of Theorem 186.

341 Theorem Let $n \in \mathbb{N}$. If A is a finite set with n elements, then the power set of A has 2^n different elements, i.e., A has 2^n different subsets.

Proof A has exactly $1 = \binom{n}{0}$ subset with 0 elements, exactly $n = \binom{n}{1}$ subsets with 1 elements,..., and exactly $1 = \binom{n}{n}$ subset with n elements. By the Binomial Theorem,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.$$

342 Example (AIME 1989) Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

Solution: Choosing k points $3 \leq k \leq 10$ points will determine a k-sided polygon, since the polygons are convex and thus have no folds. The answer is thus

$$\sum_{k=3}^{10} \binom{10}{k} = 2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} = 1024 - 1 - 10 - 45 = 968.$$

We will now derive some identities for later use.

343 Lemma

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}.$$

344 Lemma

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2}.$$

Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)}{k(k-1)} \cdot \frac{(n-2)!}{(k-2)!(n-k)!} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \binom{n-1}{k-1}.$$

345 Theorem

$$\sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = np.$$

Proof We use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} k\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n-1} n\binom{n-1}{k}p^{k+1}(1-p)^{n-1-k} \\
&= np\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k} \\
&= np(p+1-p)^{n-1} \\
&= np.
\end{aligned}$$

346 Lemma

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} = n(n-1)p^2.$$

Proof We use the identity

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

Then

$$\begin{aligned}
\sum_{k=2}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n-2} n(n-1)\binom{n-2}{k}p^{k+2}(1-p)^{n-1-k} \\
&= n(n-1)p^2\sum_{k=0}^{n-2}\binom{n-1}{k}p^k(1-p)^{n-2-k} \\
&= n(n-1)p^2(p+1-p)^{n-2} \\
&= n(n-1)p^2.
\end{aligned}$$

347 Theorem

$$\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k} = np(1-p).$$

Proof We use the identity

$$(k-np)^2 = k^2 - 2knp + n^2p^2 = k(k-1) + k(1-2np) + n^2p^2.$$

Then

$$\begin{aligned}
\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=0}^{n}(k(k-1)+k(1-2np) \\
&\quad +n^2p^2)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n}k(k-1)\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +(1-2np)\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +n^2p^2\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k} \\
&= n(n-1)p^2 + np(1-2np) + n^2p^2 \\
&= np(1-p).
\end{aligned}$$

Homework

348 Problem Expand $(a-2b)^5$.

349 Problem Expand $(2a+3b)^4$.

350 Problem By alternately putting $x = 1$ and $x = -1$ in 2.1 and adding and subtracting the corresponding quantities, deduce the identities

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots,$$

$$2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

2.9 Miscellaneous Counting Problems

351 Example n equally spaced points $1, 2, \ldots, n$ are marked on a circumference. If 15 directly opposite to 49, how many points are there total?

Solution: Points $16, 17, \ldots, 48$ are 33 in total and are on the same side of the diameter joining 15 to 49. For each of these points there is a corresponding diametrically opposite point. There are thus a total of $2 \cdot 33 + 2 = 68$ points.

352 Example An urn has 900 chips, numbered 100 through 999. Chips are drawn at random and without replacement from the urn, and the sum of their digits is noted. What is the smallest number of chips that must be drawn in order to guarantee that at least three of these digital sums be equal?

Solution: There are 27 different sums. The sums 1 and 27 only appear once (in 100 and 999), each of the other 25 sums appears thrice. Thus if $27 + 25 + 1 = 53$ are drawn, at least 3 chips will have the same sum.

353 Example Little Dwayne has 100 cards where the integers from 1 through 100 are written. He also has an unlimited supply of cards with the signs $+$ and $=$. How many true equalities can he make, if he uses each card no more than once?

Solution: The shortest equality under the stated conditions must involve 3 numbers, and hence a maximum of 33 equalities can be achieved. The 33 equalities below shew that this maximum can be achieved.

$1+75=76$	$23+64=87$	$45+53=98$
$3+74=77$	$25+63=88$	$47+52=99$
$5+73=78$	$27+62=89$	$49+51=100$
$7+72=79$	$29+61=90$	$24+26=50$
$9+71=80$	$31+60=91$	$20+28=48$
$11+70=81$	$33+59=92$	$16+30=46$
$13+69=82$	$35+58=93$	$12+32=44$
$15+68=83$	$37+57=94$	$8+34=42$
$17+67=84$	$39+56=95$	$2+38=40$
$19+66=85$	$41+55=96$	$4+6=10$
$21+65=86$	$43+54=97$	$14+22=36$

354 Example A locker room has 100 lockers. Initially, all the lockers are closed. Person 1 enters, and opens all the lockers which are even and then leaves. Person 2 enters and opens all the lockers that have numbers which are multiples of 3, closing those lockers which were already opened, and then leaves. Person 3 enters and changes the status of the lockers (from opened to closed and vice-versa) on all the lockers which have numbers which are multiples of 4. This goes on till Person 99 enters and changes the status of locker 100. Which lockers remained closed?

Solution: Consider locker n which is initially closed. Assume n has t divisors greater than 1, $1 < d_1 < d_2 < \cdots < d_{t-1} < d_t = n$. Person

$d_1 - 1$ opens the locker, Person $d_2 - 1$ closes it, Person $d_3 - 1$ opens the locker, etc. Thus locker n remains closed if and only if n has an even number of divisors greater than 1. If we add 1 to the number of divisors, we see that locker n remains closed if and only if n has an odd number of divisors. But from (**??**) $d(n)$ will be odd if and only if n is a perfect square. Thus the lockers that remain closed are the ones with a perfect square number: lockers 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100.

355 Example Give a combinatorial interpretation of the **symmetry identity**

$$\binom{n}{k} = \binom{n}{n-k}. \tag{2.2}$$

Solution: The number of ways of choosing k balls in a bag that contains n balls is the same as the number of choosing $n - k$ balls to *remain* in the bag.

356 Example Give a combinatorial interpretation of **Newton's Identity**:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \tag{2.3}$$

for $0 \leq k \leq r \leq n$.

Solution: The sinistral side counts the number of ways of selecting r elements from a set of n, then selecting k elements from those r. The dextral side counts how many ways to select the k elements first, then select the remaining $r - k$ elements to be chosen from the remaining $n - k$ elements.

357 Example Give a combinatorial interpretation to **Pascal's identity**:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{2.4}$$

Solution: Suppose we have a bag with n red balls. The number of ways of choosing k balls is n. If we now paint one of these balls blue, the number of ways of choosing k balls is the number of ways of choosing balls if we always *include* the blue ball (and this can be done in $\binom{n-1}{k-1}$) ways, plus the number of ways of choosing k balls if we always *exclude* the blue ball (and this can be done in $\binom{n-1}{k}$ ways).

358 Example Give a combinatorial proof that for integer $n \geq 1$,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Solution: The dextral side sums

$$\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{2} + \cdots + \binom{n}{n}\binom{n}{n}.$$

By the symmetry identity, this is equivalent to summing

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}.$$

Now consider a bag with $2n$ balls, n of them red and n of them blue. The above sum is counting the number of ways of choosing 0 red balls and n blue balls, 1 red ball and $n-1$ blue balls, 2 red balls and $n-2$ blue balls, etc.. This is clearly the number of ways of choosing n balls of either colour from the bag, which is $\binom{2n}{n}$.

359 Example (Derangements) Ten different letters are taken from their envelopes, read, and then randomly replaced in the envelopes. In how many ways can this replacing be done so that none of the letters will be in the correct envelope?

Solution: Let A_i be the property that the i-th letter is put back into the i-th envelope. We want

$$\text{card}\left(\complement A_1 \cap \complement A_2 \cap \cdots \cap \complement A_{10}\right).$$

Now, if we accommodate the i-th letter in its envelope, the remaining nine letters can be put in 9! different ways in the nine remaining envelopes, thus card $(A_i) = 9!$. Similarly card $(A_i \cap A_j) = 8!$, card $(A_i \cap A_j \cap A_k) = 7!$, etc. for unequal $i, j, k, \ldots$. Now, there are $\binom{10}{1}$ ways of choosing i, $\binom{10}{2}$ ways of choosing different pairs i, j, etc.. Since

$$\text{card}\,(A_1 \cup A_2 \cup \cdots \cup A_{10}) + \text{card}\,(\complement A_1 \cap \complement A_2 \cap \cdots + \cap \complement A_{10}) = 10!,$$

by the Inclusion-Exclusion Principle we gather that

$$\begin{aligned}\text{card}\,(\complement A_1 \cap \complement A_2 \ \cap \cdots \cap \complement A_{10}) &= 10! - \left(\binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! \right. \\ &\quad \left. + \cdots - \binom{10}{9}1! + \binom{10}{10}0!\right).\end{aligned}$$

360 Example (AIME 1993) How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy satisfy $a + d = b + c$ and $bc - ad = 93$?

Solution: Since $a + d = b + c$, we can write the four-tuple (a, b, c, d) as $(a, b, c, d) = (a, a + x, a + y, a + x + y)$, with integers $x, y, 0 < x < y$. Now, $93 = bc - ad = (a + x)(a + y) - a(a + x + y) = xy$. Thus either $(x, y) = (1, 93)$ or $(x, y) = (3, 31)$. In the first case

$$(a, b, c, d) = (a, a + 1, a + 93, a + 94)$$

is in the desired range for $1 \leq a \leq 405$. In the second case,

$$(a, b, c, d) = (a, a + 3, a + 31, a + 34)$$

is in the desired range for $1 \leq a \leq 465$. These two sets of four-tuples are disjoint, and so the sought number of four-tuples is 870.

361 Example $\mathscr{A}$ is a set of one hundred distinct natural numbers such that any triplet a, b, c of $\mathscr{A}$ (repetitions are allowed in a triplet) gives a non-obtuse triangle whose sides measure a, b, and c. Let

$S(\mathscr{A})$ be the sum of the perimeters obtained by adding all the triplets in $\mathscr{A}$. Find the smallest value of $S(\mathscr{A})$. Note: we count repetitions in the sum $S(\mathscr{A})$, thus all permutations of a triplet (a, b, c) appear in $S(\mathscr{A})$.

Solution: Let m be the largest member of the set and let n be its smallest member. Then $m \geq n + 99$ since there are 100 members in the set. If the triangle with sides n, n, m is non-obtuse then $m^2 \leq 2n^2$ from where

$$(n+99)^2 \leq 2n^2 \iff n^2-198n-99^2 \geq 0 \iff n \geq 99(1+\sqrt{2}) \iff n \geq 240.$$

If $n < 240$ the stated condition is not met since $m^2 \geq (n+99)^2 \geq 2n^2$ and the triangle with sides of length n, n, m is not obtuse. Thus the set

$$\mathscr{A} = \{240, 241, 242, \ldots, 339\}$$

achieves the required minimum. There are $100^3 = 1000000$ triangles that can be formed with length in $\mathscr{A}$ and so 3000000 sides to be added. Of these $3000000/100 = 30000$ are 240, 30000 are 241, etc. Thus the value required is

$$30000(240 + 241 + \cdots + 339) = (30000)\left(\frac{100(240 + 339)}{2}\right) = 868500000.$$

Homework

362 Problem Prove that the sum of the digits appearing in the integers

$$1, 2, 3, \ldots, \underbrace{99\ldots9}_{n\ 9's}$$

is $\dfrac{9n10^n}{2}$.

(Hint: Pair a with $10^n - 1 - a$.)

363 Problem Give a combinatorial proof of *Vandermonde's Convolution Identity:*

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

for positive integers $r, s \geq n$.

(Hint: Consider choosing n balls from a bag of r yellow balls and s white balls.)

Chapter 3

Discrete Probability

3.1 Probability Spaces

364 Definition A *probability* $\mathbf{P}(\cdot)$ is a real valued rule defined on subsets of a sample space Ω and satisfying the following axioms:

❶ $0 \leq \mathbf{P}(A) \leq 1$ for $A \subseteq \Omega$,

❷ $\mathbf{P}(\Omega) = 1$,

❸ for a finite or infinite sequence $A_1, A_2, \ldots \subseteq \Omega$ of disjoint events,

$$\mathbf{P}(\cup A_i) = \sum_i \mathbf{P}(A_i).$$

The number $\mathbf{P}(A)$ is called the *probability* of event A.

365 Example Let $S = \{a, b, c, d\}$ be a sample space with $\mathbf{P}(a) = 3\mathbf{P}(b)$, $\mathbf{P}(b) = 3\mathbf{P}(c)$, $\mathbf{P}(c) = 3\mathbf{P}(d)$. Find the numerical value of $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, and $\mathbf{P}(d)$.

Solution: The trick is to express all probabilities in terms of a single one. We will express $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, in terms of $\mathbf{P}(d)$. We have

$$\mathbf{P}(b) = 3\mathbf{P}(c) = 3(3\mathbf{P}(d)) = 9\mathbf{P}(d),$$

$$\mathbf{P}(a) = 3\mathbf{P}(b) = 3(9\mathbf{P}(d)) = 27\mathbf{P}(d).$$

Now

$$\begin{aligned}\mathbf{P}(a)+\mathbf{P}(b)+\mathbf{P}(c)+\mathbf{P}(d)=1 \quad &\Longrightarrow \quad 27\mathbf{P}(d)+9\mathbf{P}(d)+3\mathbf{P}(d)+\mathbf{P}(d)=1\\ &\Longrightarrow \quad \mathbf{P}(d)=\frac{1}{40}.\end{aligned}$$

Whence

$$\begin{aligned}\mathbf{P}(a) = 27\mathbf{P}(d) &= \frac{27}{40},\\ \mathbf{P}(b) = 9\mathbf{P}(d) &= \frac{9}{40},\\ \mathbf{P}(a) = 3\mathbf{P}(d) &= \frac{3}{40}.\end{aligned}$$

366 Example Let $S = \{a, b, c, d\}$ be a sample space. Outcome a is 2 times as likely as outcome b; outcome b is 4 times as likely as outcome c; outcome c is 2 times as likely as outcome d. Find

$$\mathbf{P}(a), \mathbf{P}(b), \mathbf{P}(c), \mathbf{P}(d).$$

Solution: We are given that $\mathbf{P}(a) = 2\mathbf{P}(b)$, $\mathbf{P}(b) = 4\mathbf{P}(c)$, $\mathbf{P}(c) = 2\mathbf{P}(d)$. Hence

$$\mathbf{P}(b) = 4\mathbf{P}(c) = 4(2\mathbf{P}(d)) = 8\mathbf{P}(d),$$

and

$$\mathbf{P}(a) = 2\mathbf{P}(b) = 2(8\mathbf{P}(d)) = 16\mathbf{P}(d).$$

Now

$$\begin{aligned}\mathbf{P}(a)+\mathbf{P}(b)+\mathbf{P}(c)+\mathbf{P}(d)=1 \quad &\Longrightarrow \quad 16\mathbf{P}(d)+8\mathbf{P}(d)+2\mathbf{P}(d)+\mathbf{P}(d)=1\\ &\Longrightarrow \quad 27\mathbf{P}(d)=1,\end{aligned}$$

whence $\mathbf{P}(d) = \frac{1}{27}$. This yields

$$\mathbf{P}(a) = 16\mathbf{P}(d) = \frac{16}{27},$$

$$\mathbf{P}(b) = 8\mathbf{P}(d) = \frac{8}{27},$$

and

$$\mathbf{P}(c) = 2\mathbf{P}(d) = \frac{2}{27}.$$

Probabilities are numbers between 0 and 1. Attaching to an event a probability outside this range is nonsensical.

We will now deduce some results that will facilitate the calculation of probabilities in the future.

367 Theorem Let $Y \subseteq X$. Then $\mathbf{P}(X \setminus Y) = \mathbf{P}(X) - \mathbf{P}(Y)$.

Proof Clearly $X = Y \cup (X \setminus Y)$, and $Y \cap (X \setminus Y) = \varnothing$. Thus

$$\mathbf{P}(X) = \mathbf{P}(Y) + \mathbf{P}(X \setminus Y) \implies \mathbf{P}(X) - \mathbf{P}(Y) = \mathbf{P}(X \setminus Y).$$

368 Corollary (Complementary Event Rule) Let A be an event. Then

$$\mathbf{P}(\complement A) = 1 - \mathbf{P}(A).$$

Proof Since $\mathbf{P}(\Omega) = 1$, it is enough to take $X = \Omega, Y = A, X \setminus Y = \complement A$ in the preceding theorem. ❑

369 Corollary $\mathbf{P}(\varnothing) = 0$.

Proof Take $A = \varnothing$, $\complement A = \Omega$ in the preceding corollary. ❑

370 Theorem (Probabilistic two-set Inclusion-Exclusion) Let A, B be events. Then

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B).$$

Proof Observe that

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

is a decomposition of $A \cup B$ into disjoint sets. Thus

$$\mathbf{P}(A \cup B) = \mathbf{P}(A \setminus (A \cap B)) + \mathbf{P}(B \setminus (A \cap B)) + \mathbf{P}(A \cap B).$$

Since by Theorem 367 we have $\mathbf{P}(A \setminus (A \cap B)) = \mathbf{P}(A) - \mathbf{P}(A \cap B)$ and $\mathbf{P}(B \setminus (A \cap B)) = \mathbf{P}(B) - \mathbf{P}(A \cap B)$, we deduce that

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) - \mathbf{P}(A \cap B) + \mathbf{P}(B) - \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B),$$

from where the result follows. ❑

371 Example Let $\mathbf{P}(A) = 0.8$, $\mathbf{P}(B) = 0.5$ and $\mathbf{P}(A \cap B) = 0.4$. Find $\mathbf{P}(\complement A \cap \complement B)$ and $\mathbf{P}(\complement A \cup \complement B)$.

Solution: By Theorem 370,

$$\mathbf{P}(A \cup B) = 0.8 + 0.5 - 0.4 = 0.9.$$

By Corollary 368 and the De Morgan Law's,

$$\mathbf{P}(\complement A \cap \complement B) = \mathbf{P}(\complement(A \cup B)) = 1 - \mathbf{P}(A \cup B) = 1 - 0.9 = 0.1,$$

$$\mathbf{P}(\complement A \cup \complement B) = \mathbf{P}(\complement(A \cap B)) = 1 - \mathbf{P}(A \cap B) = 1 - 0.4 = 0.5.$$

372 Example Let $\mathbf{P}(A) = 0.9$, $\mathbf{P}(B) = 0.6$. Find the maximum and minimum possible values for $\mathbf{P}(A \cap B)$.

Solution: The maximum is 0.6, it occurs when $B \subset A$. Now by Theorem 370 and using the fact that $\mathbf{P}(A \cup B) \leq 1$, we have

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B) \geq 1.5 - 1 = 0.5,$$

whence the minimum value is 0.5.

In the manner of proving Theorem 142 we may prove

373 Theorem (Probabilistic three-set Inclusion-Exclusion)

$$\begin{aligned} \mathbf{P}(A_1 \cup A_2 \cup A_3) &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) \\ &\quad - \mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_2 \cap A_3) - \mathbf{P}(A_3 \cap A_1) \\ &\quad + \mathbf{P}(A_1 \cap A_2 \cap A_3). \end{aligned}$$

374 Definition A *random variable* **X** is a rule that to each outcome point of the sample space (the inputs) assigns a real number output. This output is not fixed, but assigned with a certain probability. The *range* or *image* of **X** is the set of outputs assumed by **X**.

375 Example A fair die is tossed. If the resulting number is even, you add 1 to your score and get that many dollars. If the resulting number is odd, you add 2 to your score and get that many dollars. Let **X** be the random variable counting your gain, in dollars. Then the range of **X** is $\{3, 5, 7\}$.

376 Example A hand of three cards is chosen from a standard deck of cards. You get $\$3$ for each heart in your hand. Let **Z** be the random variable measuring your gain. Then the range of **Z** is $\{0, 3, 6, 9\}$.

Homework

377 Problem Let $S = \{a, b, c, d\}$ be a sample space. Outcome a is 5 times as likely as outcome b; outcome b it 5 times as likely as event c; outcome c it 5 times as likely as event d. Find $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, $\mathbf{P}(d)$.

378 Problem Let $S = \{a, b, c, d\}$ be a probabilistic outcome space. It is known that outcome d is twice as likely as outcome c, outcome c is four times as likely as outcome b, and outcome b is half as likely as outcome a. Find $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, $\mathbf{P}(d)$.

379 Problem (AHSME 1983) It is known that $\mathbf{P}(A) = \frac{3}{4}$ and $\mathbf{P}(B) = \frac{2}{3}$. Shew that $\frac{5}{12} \leq \mathbf{P}(A \cap B) \leq \frac{2}{3}$.

380 Problem Three fair dice, a red, a white and a blue one are thrown. The **sum** of the dots is given by the random variable **Y**. What is the range of the random variable **Y**?

381 Problem Two fair dice, a red and a blue one are thrown. The **product** of the dots is given by the random variable **Y**. What is the range of the random variable **Y**?

382 Problem A fair die is tossed. If the resulting number is either 2 or 3, you multiply your score by 2 and get that many dollars. If the resulting number is either 1 or 4, you add 1 to your score and get that many dollars. If the resulting number is either 5 or 6, you get that many dollars. Let **X** be the random variable counting your gain, in dollars. Give the range of **X**.

383 Problem There are two telephone lines A and B. Let E_1 be the event that line A is engaged and let E_2 be the event that line B is engaged. After a statistical study one finds that $\mathbf{P}(E_1) = 0.5$ and $\mathbf{P}(E_2) = 0.6$ and $\mathbf{P}(E_1 \cap E_2) = 0.3$. Find the probability of the following events:

❶ F: "line A is free."

❷ G: "at least one line is engaged."

❸ H: "at most one line is free."

Answer: 0.5, 0.8, 0.7

384 Problem For events A and B you are given that $\mathbf{P}(A) = \frac{1}{3}$, $\mathbf{P}(B) = \frac{2}{5}$, and $\mathbf{P}(A \cup B) = \frac{3}{4}$. Find $\mathbf{P}(\complement A)$, $\mathbf{P}(\complement B)$, $\mathbf{P}(A \cap B)$, $\mathbf{P}(\complement A \cup \complement B)$, $\mathbf{P}(\complement A \cap \complement B)$.

3.2 Uniform Random Variables

Consider a non-empty finite set Ω with $\text{card}(\Omega)$ number of elements and let A, B be disjoint subsets of Ω. It is clear that

❶ $0 \leq \dfrac{\text{card}(A)}{\text{card}(\Omega)} \leq 1$,

❷ $\dfrac{\text{card}(\Omega)}{\text{card}(\Omega)} = 1$,

❸ $\dfrac{\text{card}(A \cup B)}{\text{card}(\Omega)} = \dfrac{\text{card}(A)}{\text{card}(\Omega)} + \dfrac{\text{card}(B)}{\text{card}(\Omega)}$.

Thus the quantity $\dfrac{\text{card}(A)}{\text{card}(\Omega)}$ on the subsets of Ω is a probability (satisfies definition 364), and we put

$$\mathbf{P}(A) = \frac{\text{card}(A)}{\text{card}(\Omega)}. \tag{3.1}$$

Observe that in this model the probability of any single outcome is $\dfrac{1}{\text{card}(\Omega)}$, that is, every outcome is *equally likely*.

385 Definition Let

$$\Omega = \{x_1, x_2, \ldots, x_n\}$$

be a finite sample space. A *uniform discrete random variable* $\mathbf{X}$ *defined on* Ω is a function that achieves the distinct values x_k with equal probability:

$$\mathbf{P}(\mathbf{X} = x_k) = \frac{1}{\text{card}(\Omega)}.$$

Since

$$\sum_{k=1}^{n} \mathbf{P}(\mathbf{X} = x_k) = \sum_{k=1}^{n} \frac{1}{\text{card}(\Omega)} = \frac{\text{card}(\Omega)}{\text{card}(\Omega)} = 1,$$

this is a bonafide random variable.

386 Example If the experiment is flipping a fair coin, then $\Omega = \{H, T\}$ is the sample space (H for heads, T for tails) and $E = \{H\}$ is the event of obtaining a head. Then

$$\mathbf{P}(H) = \frac{1}{2} = \mathbf{P}(T).$$

387 Example If the experiment is rolling a red fair die and a blue fair die and then adding their scores, the sample space consists of $6 \cdot 6 = 36$ possible outcomes. If **S** denotes the random variable of the sum obtained then $2 \leq \mathbf{S} \leq 12$. These sums are obtained in the following fashion:

S	(red, blue)
2	$(1,1)$
3	$(1,2),(2,1)$
4	$(1,3),(3,1),(2,2)$
5	$(1,4),(4,1),(2,3),(3,2)$
6	$(1,5),(5,1),(2,4),(4,2),(3,3)$
7	$(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)$
8	$(2,6),(6,2),(3,5),(5,3),(4,4)$
9	$(3,6),(6,3),(4,5),(5,4)$
10	$(4,6),(6,4),(5,5)$
11	$(5,6),(6,5)$
12	$(6,6)$

Therefore

$$\mathbf{P}(\mathbf{S}=2)=\frac{1}{36},$$
$$\mathbf{P}(\mathbf{S}=3)=\frac{2}{36}=\frac{1}{18},$$
$$\mathbf{P}(\mathbf{S}=4)=\frac{3}{36}=\frac{1}{12},$$
$$\mathbf{P}(\mathbf{S}=5)=\frac{4}{36}=\frac{1}{9},$$
$$\mathbf{P}(\mathbf{S}=6)=\frac{5}{36},$$
$$\mathbf{P}(\mathbf{S}=7)=\frac{6}{36}=\frac{1}{6},$$
$$\mathbf{P}(\mathbf{S}=8)=\frac{5}{36},$$
$$\mathbf{P}(\mathbf{S}=9)=\frac{4}{36}=\frac{1}{9},$$
$$\mathbf{P}(\mathbf{S}=10)=\frac{3}{36}=\frac{1}{12},$$
$$\mathbf{P}(\mathbf{S}=11)=\frac{2}{36}=\frac{1}{18},$$
$$\mathbf{P}(\mathbf{S}=12)=\frac{1}{36}.$$

STOP *In a fair die there are $7-x$ dots on the face opposite x dots. Hence $\mathbf{P}(\mathbf{S}=x)=\mathbf{P}(\mathbf{S}=14-x)$.*

388 Example A number X is chosen at random from the set $\{1,2,\ldots,25\}$. Find the probability that when divided by 6 it leaves remainder 1.

Solution: There are only 4 numbers in the set that leave remainder 1 upon division by 6, namely $\{7,13,19,25\}$. The probability sought is thus $\frac{4}{25}$.

389 Example A number is chosen at random from the set

$$\{1, 2, \ldots, 1000\}.$$

What is the probability that it is a palindrome?

Solution: There are 9 palindromes with 1-digit, 9 with 2 digits and 90 with three digits. Thus the number of palindromes in the set is $9 + 9 + 90 = 108$. The probability sought is $\dfrac{108}{1000} = \dfrac{27}{250}$.

390 Example A fair die is rolled three times and the scores added. What is the probability that the sum of the scores is 6?

Solution: Let A be the event of obtaining a sum of 6 in three rolls, and let Ω be the sample space created when rolling a die thrice. The sample space has $6^3 = 216$ elements, since the first roll can land in 6 different ways, as can the second and third roll. To obtain a sum of 6 in three rolls, the die must have the following outcomes:

$$\begin{aligned} A &= \{(2,2,2),(4,1,1),(1,4,1),(1,1,4),(1,2,3),\\ &\quad (1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\} \end{aligned}$$

and so $\text{card}\,(A) = 10$. Hence $\mathbf{P}\,(A) = \dfrac{10}{216} = \dfrac{5}{108}$.

391 Example Consider a standard deck of cards. One card is drawn at random.

❶ Find the size of the sample space of this experiment.

❷ Find the probability $\mathbf{P}\,(K)$ of drawing a king.

❸ Find the probability $\mathbf{P}\,(J)$ of drawing a knave[1].

❹ Find the probability $\mathbf{P}\,(R)$ of drawing a red card.

[1] A *knave* is what vulgar people call a *jack*. Cf. Charles Dickens' *Great Expectations*.

❺ Find the probability $\mathbf{P}(K \cap R)$ of drawing a red king.

❻ Find the probability $\mathbf{P}(K \cup R)$ of drawing either a king or a red card.

❼ Find the probability $\mathbf{P}(K \setminus R)$ of drawing a king which is not red.

❽ Find the probability $\mathbf{P}(R \setminus K)$ of drawing a red card which is not a king.

❾ Find the probability $\mathbf{P}(K \cap J)$ of drawing a king which is also a knave.

Solution:

❶ The size of the sample space for this experiment is $\text{card}(S) = \binom{52}{1} = 52$.

❷ Since there are 4 kings, $\text{card}(K) = 4$. Hence $\mathbf{P}(K) = \frac{4}{52} = \frac{1}{13}$.

❸ Since there are 4 knaves, $\text{card}(J) = 4$. Hence $\mathbf{P}(J) = \frac{4}{52} = \frac{1}{13}$.

❹ Since there are 26 red cards, $\text{card}(R) = 26$. Hence $\mathbf{P}(R) = \frac{26}{52} = \frac{1}{2}$.

❺ Since a card is both a king and red in only two instances (when it is $K\heartsuit$ or $K\diamondsuit$), we have $\mathbf{P}(K \cap R) = \frac{2}{52} = \frac{1}{26}$.

❻ By Inclusion-Exclusion we find

$$\mathbf{P}(K \cup R) = \mathbf{P}(K) + \mathbf{P}(R) - \mathbf{P}(K \cap R) = \frac{4 + 26 - 2}{52} = \frac{28}{52} = \frac{7}{13}.$$

❼ Since of the 4 kings two are red we have $\mathbf{P}(K \setminus R) = \frac{2}{52} = \frac{1}{26}$.

❽ Since of the 26 red cards two are kings, $\mathbf{P}(R \setminus K) = \frac{24}{52} = \frac{6}{13}$.

❾ Since no card is simultaneously a king and a knave, $\mathbf{P}(K \cap J) = \mathbf{P}(\varnothing) = 0$.

392 Example Phone numbers in a certain town are 7-digit numbers that do not start in $0, 1$, or 9. What is the probability of getting a phone number in this town that is divisible by 5?

Solution: The sample space consists of all possible phone numbers in this town: $7 \cdot 10^6$. A phone number will be divisible by 5 if it ends in 0 or 5 and so there are $7 \cdot 10^5 \cdot 2$ phone numbers that are divisible by 5. The probability sought is

$$\frac{7 \cdot 10^5 \cdot 2}{7 \cdot 10^6} = \frac{2}{10} = \frac{1}{5}.$$

393 Example Consider a standard deck of cards. Four cards are chosen at random without regards to order and without replacement. Then

❶ The sample space for this experiment has size

$$\binom{52}{4} = 270725.$$

❷ The probability of choosing the four kings is

$$\frac{\binom{4}{4}}{\binom{52}{4}} = \frac{1}{270725}.$$

❸ The probability of choosing four cards of the same face is

$$\frac{\binom{13}{1}\binom{4}{4}}{\binom{52}{4}} = \frac{13}{270725} = \frac{1}{20825}.$$

❹ The probability of choosing four cards of the same colour is

$$\frac{\binom{2}{1}\binom{26}{4}}{\binom{52}{4}} = \frac{(2)(14950)}{270725} = \frac{92}{833}.$$

❺ The probability of choosing four cards of the same suit is

$$\frac{\binom{4}{1}\binom{13}{4}}{\binom{52}{4}} = \frac{(4)(715)}{270725} = \frac{44}{4165}.$$

394 Example A hat contains 20 tickets, each with a different number from 1 to 20. If 4 tickets are drawn at random, what is the probability that the largest number is 15 and the smallest number is 9?

Solution: For this to happen, we choose the ticket numbered 9, the one numbered 15 and the other two tickets must be chosen from amongst the five tickets numbered $10, 11, 12, 13, 14$. The probability sought is thus

$$\frac{\binom{5}{2}}{\binom{20}{4}} = \frac{10}{4845} = \frac{2}{969}.$$

395 Example A box contains four \$10 bills, six \$5 bills, and two \$1 bills. Two bills are taken at random from the box without replacement. What is the probability that both bills will of the same denomination?

Solution: There are $4 + 6 + 2 = 12$ bills. The experiment can be performed in $\binom{12}{2} = 66$ ways. To be successful we must choose either 2 tens (in $\binom{4}{2} = 6$ ways), **or** 2 fives (in $\binom{6}{2} = 15$ ways), **or** 2 ones (in $\binom{2}{2} = 1$ way). The probability sought is thus

$$\frac{\binom{4}{2} + \binom{6}{2} + \binom{2}{2}}{\binom{12}{2}} = \frac{6 + 15 + 1}{66} = \frac{1}{3}.$$

396 Example From a group of A males and B females a committee of C people will be chosen.

❶ What is the probability that there are exactly T females?

❷ What is the probability that at most 3 females will be chosen?

❸ What is the probability that Mary and Peter will be serving together in a committee?

❹ What is the probability that Mary and Peter will not be serving together?

Solution:

❶ First observe that this experiment has a sample space of size $\binom{A+B}{C}$. There are $\binom{B}{T}$ ways of choosing the females. The remaining $C-T$ members of the committee must be male, hence the desired probability is

$$\frac{\binom{B}{T}\binom{A}{C-T}}{\binom{A+B}{C}}.$$

❷ Either $C-2$ or $C-1$ or C males will be chosen. Corresponding to each case, we must choose either 2 or 1 or 0 women, whence the desired probability is

$$\frac{\binom{B}{C-2}\binom{A}{2}+\binom{B}{C-1}\binom{A}{1}+\binom{B}{C}\binom{A}{0}}{\binom{A+B}{C}}.$$

❸ Either 3 or 2 or 1 or 0 women will be chosen. In each case, either $C-3$ or $C-2$ or $C-1$ or C men will be chosen. Thus the desired probability is

$$\frac{\binom{A}{C-3}\binom{B}{3}+\binom{A}{C-2}\binom{B}{2}+\binom{A}{C-1}\binom{B}{1}+\binom{A}{C}\binom{B}{0}}{\binom{A+B}{C}}.$$

❹ We must assume that Peter and Mary belong to the original set of people, otherwise the probability will be 0. Since Peter and Mary must belong to the committee, we must choose $C-2$ other people from the pool of the $A+B-2$ people remaining. The desired probability is thus

$$\frac{\binom{A+B-2}{C-2}}{\binom{A+B}{C}}.$$

❺ Again, we must assume that Peter and Mary belong to the original set of people, otherwise the probability will be 1. Observe that one of the following three situations may arise: (1) Peter is in a committee, Mary is not, (ii) Mary is in a committee, Peter is not, (iii) Neither Peter nor Mary are in a committee. Perhaps the easiest way to count these options (there are many ways of doing this) is to take the total number of committees and subtract those including (simultaneously) Peter and Mary. The desired probability is thus

$$\frac{\binom{A+B}{C}-\binom{A+B-2}{C-2}}{\binom{A+B}{C}}.$$

Aliter: The number of committees that include Peter but exclude Mary is $\binom{A+B-2}{C-1}$, the number of committees that include Mary but exclude Peter is $\binom{A+B-2}{C-1}$, and the number of committees that exclude both Peter and Mary is $\binom{A+B-2}{C}$. Thus the desired probability is seen to be

$$\frac{\binom{A+B-2}{C-1}+\binom{A+B-2}{C-1}+\binom{A+B-2}{C}}{\binom{A+B}{C}}$$

That this agrees with the preceding derivation is a simple algebraic exercise.

397 Example A number X is chosen at random from the series

$$2, 5, 8, 11 \ldots, 299$$

and another number Y is chosen from the series

$$3, 7, 11, \ldots, 399.$$

What is the probability $\mathbf{P}(X = Y)$?

Solution: There are 100 terms in each of the arithmetic progressions. Hence we may choose X in 100 ways and Y in 100 ways. The size of the sample space for this experiment is thus $100 \cdot 100 = 10000$. Now we note that 11 is the smallest number that belongs to both progressions. Since the first progression has common difference 3 and the second progression has common difference 4, and since the least common multiple of 3 and 14 is 12, the progressions have in common numbers of the form

$$11 + 12k.$$

We need

$$11 + 12k \leq 299 \implies k = 24.$$

Therefore, the 25 numbers

$$11 = 11 + 12 \cdot 0, 23 = 11 + 12 \cdot 1, 35 = 11 + 12 \cdot 2, \ldots, 299 = 11 + 12 \cdot 24$$

belong to both progressions and the probability sought is

$$\frac{25}{10000} = \frac{1}{400}.$$

398 Example A number N is chosen at random from $\{1, 2, \ldots, 25\}$. Find the probability that $N^2 + 1$ be divisible by 10.

Solution: $N^2 + 1$ is divisible by 10 if it ends in 0. For that N^2 must end in 9. This happens when $N \in \{3, 7, 13, 17, 23\}$. Thus the probability sought is $\frac{5}{25} = \frac{1}{5}$.

399 Example (Poker Hands) A poker hand consists of 5 cards from a standard deck of 52 cards, and so there are $\binom{52}{5} = 2598960$ ways of selecting a poker hand. Various hands, and their numbers, are shewn below.

❶ **1 pair** occurs when you have one pair of faces of any suit, and none of the other faces match. For example, $A\clubsuit, A\diamondsuit, 2\heartsuit, 4\clubsuit, 6\diamondsuit$ is a pair. The number of ways of getting a pair is

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1098240$$

and so the probability of getting a pair is $\frac{1098240}{2598960} \approx 0.422569$.

❷ **2 pairs** occurs when you have 2 *different* pairs of faces of any suit, and the remaining card of a different face than the two pairs. For example, $A\clubsuit, A\diamondsuit, 3\heartsuit, 3\diamondsuit, 7\heartsuit$ is a 2 pair. The number of ways of getting two pairs is $\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123552$ and so the probability of getting 2 pairs is $\frac{123552}{2598960} \approx 0.047539$.

❸ **3 of a kind** occurs when you have three cards of the same face and the other two cards are from a different face. For example, $A\clubsuit, A\diamondsuit, A\spadesuit, 3\spadesuit, 7\diamondsuit$. The number of ways of getting a 3 of a kind is $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54912$ and so the probability of this event is $\frac{54912}{2598960} \approx 0.021128$.

❹ **straight** occurs when the faces are consecutive, but no four cards belong to the same suit, as in $2\clubsuit, 3\heartsuit, 4\spadesuit, 5\spadesuit, 6\diamondsuit$. The number of ways of getting a straight is $10(4^5 - 4) = 10200$ and so the probability of this event is $\frac{10200}{2598960} \approx 0.003925$.

❺ **flush** occurs when you have five non-consecutive cards of the same suit, as in $2\clubsuit, 4\clubsuit, 7\clubsuit, 8\clubsuit, 10\clubsuit$. The number of ways of obtaining a flush is $\binom{4}{1}\binom{13}{5} = 5108$ and so the probability of this event is $\frac{5108}{2598960} \approx 0.001965$.

❻ **full house** occurs when 3 cards have the same face and the other two cards have the same face (different from the first three cards), as in $8\clubsuit, 8\spadesuit, 8\diamondsuit, 7\heartsuit, 7\clubsuit$. The number of ways of getting this is $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3774$ and so the probability of this event is $\frac{3774}{2598960} \approx 0.001441$.

❼ **4 of a kind** occurs when a face appears four times, as in $8\clubsuit, 8\spadesuit, 8\diamondsuit, 8\heartsuit, 7\clubsuit$. The number of ways of getting this is

$$\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1} = 624,$$

and the probability for this event is $\frac{624}{2598960} \approx 0.00024$.

❽ **straight flush** occurs when one gets five consecutive cards of the same suit, as in $2♣, 3♣, 4♣, 5♣, 6♣$. The number of ways of getting this is $\binom{4}{1}10 = 40$, and the probability of this event is $\frac{40}{2598960} \approx 0.000015$.

400 Example (The Birthday Problem) If there are n people in a classroom, what is the probability that no pair of them celebrates their birthday on the same day of the year?

Solution: To simplify assumptions, let us discard 29 February as a possible birthday and let us assume that a year has 365 days. There are 365^n n-tuples, each slot being the possibility of a day of the year for each person. The number of ways in which no two people have the same birthday is

$$365 \cdot 364 \cdot 363 \cdots (365 - n + 1),$$

as the first person can have his birthday in 365 days, the second in 364 days, etc. Thus if A is the event that no two people have the same birthday, then

$$\mathbf{P}(A) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}.$$

The probability sought is

$$\mathbf{P}(\complement A) = 1 - \mathbf{P}(A) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}.$$

A numerical computation shews that for $n = 23$, $\mathbf{P}(A) < \frac{1}{2}$, and so $\mathbf{P}(\complement A) > \frac{1}{2}$. This means that if there are 23 people in a room, the probability is better than $\frac{1}{2}$ that two will have the same birthday.

401 Example Three fair dice, a red, a white, and a blue one are tossed, and their scores registered in the random variables R, W, B respectively. What is the probability that $R \leq W \leq B$?

Solution: Each of the dice may land in 6 ways and hence the size of the sample space for this experiment is $6^3 = 216$. Notice that there is a one to one correspondence between vectors

$$(R, W, B), \quad 1 \leq R \leq W \leq B \leq 6$$

and vectors

$$(R', W', B'), \quad 1 \leq R' < W' < B' \leq 8.$$

This can be seen by putting $R' = R, W' = W + 1$, and $B' = B + 2$. Thus the number of vectors (R', W', B') with $1 \leq R' < W' < B' \leq 8$ is $\binom{8}{3} = 56$. The probability sought is thus

$$\frac{56}{216} = \frac{7}{27}.$$

402 Example A hat contains three tickets, numbered $1, 2$ and 3. The tickets are drawn from the box one at a time. Find the probability that the ordinal number of at least one ticket coincides with its own number.

Solution: Let $A_k, k = 1, 2, 3$ be the event that when drawn from the hat, ticket k is the k-th chosen. We want

$$\mathbf{P}(A_1 \cup A_2 \cup A_3).$$

By inclusion-exclusion for three sets Theorem 373

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) \\ &\quad -\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_2 \cap A_3) - \mathbf{P}(A_3 \cap A_1) \\ &\quad +\mathbf{P}(A_1 \cap A_2 \cap A_3)\end{aligned}$$

By symmetry,

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \mathbf{P}(A_3) = \frac{2!}{3!} = \frac{1}{3},$$

$$\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_2 \cap A_3) = \mathbf{P}(A_3 \cap A_1) = \;= \frac{1!}{3!} = \frac{1}{6},$$

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{3!} = \frac{1}{6}.$$

The probability sought is finally

$$\mathbf{P}(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

403 Example (AHSME 1994) When n standard six-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. What is the smallest possible value of S?

Solution: Since the probability of obtaining the sum 1994 is positive, there are $n \geq \lfloor \frac{1994}{6} \rfloor = 333$ dice. Let $x_1 + x_2 + \cdots + x_n = 1994$ be the sum of the faces of the n dice adding to 1994. We are given that

$$(7 - x_1) + (7 - x_2) + \cdots + (7 - x_n) = S$$

or $7n - 1994 = S$. The minimal sum will be achieved with the minimum dice, so putting $n = 333$ we obtain the minimal $S = 7(333) - 1994 = 337$.

Homework

404 Problem There are 100 cards: 10 of each red—numbered 1 through 10; 20 white—numbered 1 through 20; 30 blue—numbered 1 through 30; and 40 magenta—numbered 1 through 40.

❶ Let R be the event of picking a red card. Find $\mathbf{P}(R)$.

❷ Let B be the event of picking a blue card. Find $\mathbf{P}(B)$.

❸ Let E be the event of picking a card with face value 11. Find $\mathbf{P}(E)$.

❹ Find $\mathbf{P}(B \cup R)$.

❺ Find $\mathbf{P}(E \cap R)$.

❻ Find $\mathbf{P}(E \cap B)$.

❼ Find $\mathbf{P}(E \cup R)$.

❽ Find $\mathbf{P}(E \cup B)$.

❾ Find $\mathbf{P}(E \setminus B)$.

❿ Find $\mathbf{P}(B \setminus E)$.

Answer: $\frac{1}{10}; \frac{3}{10}; \frac{3}{100}; \frac{2}{5}; 0; \frac{1}{100}; \frac{13}{100}; \frac{8}{25}; \frac{1}{50}; \frac{29}{100}$

405 Problem Find the chance of throwing at least one ace in a single throw of two dice.

Answer: $\frac{11}{36}$

406 Problem An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn at once from the urn, and their colour noted. What is the probability that a marble of each colour is drawn?

Answer: $\frac{\binom{3}{1}\binom{4}{1}\binom{5}{1}}{\binom{12}{3}} = \frac{3}{11}$

407 Problem **One** card is drawn at random from a standard deck. What is the probability that it is a queen?

Answer: $\frac{1}{13}$

408 Problem **Two** cards are drawn at random from a standard deck. What is the probability that both are queens?

Answer: $\frac{1}{221}$

409 Problem **Four** cards are drawn at random from a standard deck. What is the probability that two are red queens and two are spades?

Answer: $\frac{6}{20825}$

410 Problem **Four** cards are drawn at random from a standard deck. What is the probability that there are no hearts?

Answer: $\frac{6327}{20825}$

411 Problem A $3 \times 3 \times 3$ wooden cube is painted red and cut into 27 $1 \times 1 \times 1$ smaller cubes. These cubes are mixed in a hat and one of them chosen at random. What is the probability that it has exactly 2 of its sides painted red?

Answer: $\frac{4}{9}$

412 Problem Three fair dice are thrown at random.

❶ Find the probability of getting at least one 5 on the faces.

❷ Find the probability of obtaining at least two faces with the same number.

❸ Find the probability that the sum of the points on the faces is even.

Answer: $\frac{125}{216}; \frac{4}{9}; \frac{1}{2}$

413 Problem Six cards are drawn without replacement from a standard deck of cards. What is the probability that

❶ three are red and three are black?

❷ two are queens, two are aces, and two are kings?

❸ four have the same face (number or letter)?

❹ exactly four are from the same suit?

❺ there are no queens?

Answer: $\dfrac{\binom{26}{3}^2}{\binom{52}{6}}$; $\dfrac{\binom{4}{2}^3}{\binom{52}{6}}$; $\dfrac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{6}}$; $\dfrac{\binom{4}{1}\binom{13}{4}\binom{39}{2}}{\binom{52}{6}}$; $\dfrac{\binom{48}{6}}{\binom{52}{6}}$

414 Problem An ordinary fair die and a die whose faces have $2, 3, 4, 6, 7, 9$ dots but is otherwise balanced are tossed and the total noted. What is the probability that the sum of the dots shewing on the dice exceeds 9?

Answer: $\dfrac{7}{18}$

415 Problem Let k, N be positive integers. Find the probability that an integer chosen at random from $\{1, 2, \ldots, N\}$ be divisible by k.

Answer: $\dfrac{\lfloor \frac{N}{k} \rfloor}{N}$

416 Problem What is the probability that a random integer taken from $\{1, 2, 3, \ldots, 100\}$ has no factors in common with 100?

Answer: $\dfrac{2}{5}$

417 Problem A number N is chosen at random from $\{1, 2, \ldots, 25\}$. Find the probability that $N^2 - 1$ be divisible by 10.

Answer: $\dfrac{1}{5}$

418 Problem Two different numbers X and Y are chosen from $\{1, 2, \ldots, 10\}$. Find the probability that $XY \leq 27$.

419 Problem Two different numbers X and Y are chosen from $\{1, 2, \ldots, 10\}$. Find the probability that $X^2 + Y^2 \leq 27$.

420 Problem Two different numbers X and Y are chosen from $\{1, 2, \ldots, 10\}$. Find the probability that XY is a perfect square.

421 Problem Two different numbers X and Y are chosen from $\{1, 2, \ldots, 10\}$. Find the probability that X and Y are relatively prime.

422 Problem Ten different numbers are chosen at random from the set of 30 integers $\{1, 2, \ldots, 30\}$. Find the probability that

❶ all the numbers are odd.

❷ exactly 5 numbers be divisible by 3.

❸ exactly 5 numbers are even, and exactly one of them is divisible by 10.

423 Problem There are two winning tickets amongst ten tickets available. Determine the probability that (a) one, (b) both tickets will be among five tickets selected at random.

Answer: $\frac{5}{9}; \frac{2}{9}$

424 Problem Find the chance of throwing more that 15 in a single throw of three dice.

Answer: $\frac{5}{108}$

425 Problem Little Edna is playing with the four letters of her name, arranging them at random in a row. What is the probability that the two vowels come together?

Answer: $\frac{1}{4}$

426 Problem (Galileo's Paradox) Three distinguishable fair dice are thrown (say, one red, one blue, and one white). Observe that

$$9 = 1+2+6 = 1+3+5 = 1+4+4 = 2+2+5 = 2+3+4 = 3+3+3,$$

and

$$10 = 1+3+6 = 1+4+5 = 2+2+6 = 2+3+5 = 2+4+4 = 3+3+4.$$

The probability that a sum $\mathbf{S}$ of 9 appears is lower than the probability that a sum of 10 appears. Explain why and find these probabilities.

Answer: $\mathbf{P}(\mathbf{S}=9) = \frac{25}{216} \approx 0.1157$, $\mathbf{P}(\mathbf{S}=10) = \frac{1}{8} = 0.125$

427 Problem Five people entered the lift cabin on the ground floor of an 8-floor building (this includes the ground floor). Suppose each of them, independently and with equal probability, can leave the cabin at any of the other seven floors. Find out the probability of all five people leaving at different floors.

Answer: $\frac{360}{2401}$

428 Problem (AHSME 1984) A box contains 11 balls, numbered $1, 2, \ldots 11$. If six balls are drawn simultaneously at random, find the probability that the sum of the numbers on the balls drawn is odd.

Answer: $\frac{118}{231}$

429 Problem A hat contains 7 tickets numbered 1 through 7. Three are chosen at random. What is the probability that their product be an odd integer?

Answer: $\frac{\binom{4}{3}}{\binom{7}{3}} = \frac{4}{35}$

430 Problem (AHSME 1986) Six distinct integers are chosen at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

Answer: $\dfrac{1}{3}$

431 Problem An urn contains n black and n white balls. Three balls are chosen from the urn at random and without replacement. What is the value of n if the probability is $\dfrac{1}{12}$ that all three balls are white?

Answer: 5

432 Problem A fair die is tossed twice in succession. Let **A** denote the first score and **B** the second score. Consider the quadratic equation

$$x^2 + \mathbf{A}x + \mathbf{B} = 0.$$

Find the probability that

❶ the equation has 2 distinct roots.

❷ the equation has a double root.

❸ $x = -3$ be a root of the equation,

❹ $x = 3$ be a root of the equation.

Answer: $\dfrac{17}{36}$; $\dfrac{1}{18}$; $\dfrac{1}{18}$; 0

433 Problem An urn contains $3n$ counters: n red, numbered 1 through n, n white, numbered 1 through n, and n blue, numbered 1 through n. Two counters are to be drawn at random without replacement. What is the probability that both counters will be of the same colour or bear the same number?

Answer: $\dfrac{n+1}{3n-1}$

434 Problem (AIME 1984) A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to each other. Find $m+n$.

Answer: 106

435 Problem Five fair dice are thrown. What is the probability that a full house in thrown (that is, where two dice shew one number and the other three dice shew a second number)?

Answer: $\frac{25}{648}$

436 Problem If thirteen cards are randomly chosen without replacement from an ordinary deck of cards, what is the probability of obtaining exactly three aces?

Answer: $\frac{858}{20825}$

437 Problem A calculator has a random number generator button which, when pushed displays a random digit $\{0,1,2,3,4,5,6,7,8,9\}$. The button is pushed four times. Assuming the numbers generated are independent, what is the probability of obtaining one '0', one '5', and two '9's in any order?

Answer: $\frac{3}{2500}$

438 Problem Mrs. Flowers plants rosebushes in a row. Eight of the bushes are white and two are red, and she plants them in a random order. What is the probability that she will consecutively plants seven or more white bushes?

Answer: $\frac{1}{5}$

439 Problem All the letters from a sign marked

OHIO

USA

have fallen down. Little Punctus Gallicus, who is friendly but illiterate, puts the letters back haphazardly, four in the first row and three in the second. She is equally likely to put the letter correctly or upside down.

❶ What is the probability that will she spell the first word correctly?

❷ What is the probability that will she spell the second word correctly?

Answer: $\frac{1}{420}, \frac{1}{1680}$

440 Problem (MMPC 1992) From the set $\{1, 2, \ldots, n\}$, k distinct integers are selected at random and arranged in numerical order (lowest to highest). Let $\mathbf{P}(i, r, k, n)$ denote the probability that integer i is in position r. For example, observe that $\mathbf{P}(1, 2, k, n) = 0$ and $\mathbf{P}(2, 1, 6, 10) = 4/15$. Find a general formula for $\mathbf{P}(i, r, k, n)$.

Answer: $\mathbf{P}(i, r, k, n) = \frac{\binom{i-1}{r-1}\binom{n-i}{k-r}}{\binom{n}{k}}$

3.3 Independence

441 Definition Two events A and B are said to be independent if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B).$$

442 Example Let A, B be independent events with $\mathbf{P}(A) = \mathbf{P}(B)$ and $\mathbf{P}(A \cup B) = \frac{1}{2}$. Find $\mathbf{P}(A)$.

Solution: By inclusion-exclusion Theorem 370,

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B),$$

which yields

$$\frac{1}{2} = 2\mathbf{P}(A) - (\mathbf{P}(A))^2 \implies 2x^2 - 4x + 1 = 0,$$

with $x = \mathbf{P}(A)$. Solving this quadratic equation and bearing in mind that we must have $0 < x < 1$, we find $\mathbf{P}(A) = x = 1 - \frac{\sqrt{2}}{2}$.

More often than not independence is built into a problem physically, that is, an event A does not physically influence an event B.

443 Example Two dice, a red one and a blue one, are thrown. If A is the event: "the red die lands on an even number" and B is the event: "the blue die lands on a prime number" then A and B are independent, as they do not physically influence one another.

444 Example A die is loaded so that if $\mathbf{D}$ is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{D} = k) = \frac{k}{21}$, where $k = 1, 2, 3, 4, 5, 6$. Another die is loaded differently, so that if $\mathbf{X}$ is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{X} = k) = \frac{k^2}{91}$. Find $\mathbf{P}(\mathbf{D} + \mathbf{X} = 4)$.

Solution: Clearly the value on which the first die lands does not influence the value on which the second die lands. Thus by independence

$$\begin{aligned}
\mathbf{P}(\mathbf{D}+\mathbf{X}=4) &\iff \mathbf{P}(\mathbf{D}=1\cap\mathbf{X}=3)+\mathbf{P}(\mathbf{D}=2\cap\mathbf{X}=2)\\
&\quad +\mathbf{P}(\mathbf{D}=3\cap\mathbf{X}=1)\\
&= \mathbf{P}(\mathbf{D}=1)\cdot\mathbf{P}(\mathbf{X}=3)+\mathbf{P}(\mathbf{D}=2)\cdot\mathbf{P}(\mathbf{X}=2)\\
&\quad +\mathbf{P}(\mathbf{D}=3)\cdot\mathbf{P}(\mathbf{X}=1)\\
&= \frac{1}{91}\cdot\frac{3}{21}+\frac{4}{91}\cdot\frac{2}{21}+\frac{9}{91}\cdot\frac{1}{21}\\
&= \frac{20}{1911}.
\end{aligned}$$

When we deal with more than two events, the following definition is pertinent.

445 Definition The events $A_1, A_2, \ldots, A_n$ are independent if for any choice of k $(2 \leq k \leq n)$ indexes $\{i_1, i_2, \ldots, l_k\}$ we have

$$\mathbf{P}(A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}) = \mathbf{P}(A_{i_1})\,\mathbf{P}(A_{i_2})\cdots\mathbf{P}(A_{i_k}).$$

Considerations of independence are important in the particular case when trials are done in succession.

446 Example A biased coin with $\mathbf{P}(H) = \frac{2}{5}$ is tossed three times in a row. Find the probability that one will obtain HHT, in that order.

Solution: Each toss is physically independent from the other. The required probability is

$$\mathbf{P}(HHT) = \mathbf{P}(H)\cdot\mathbf{P}(H)\cdot\mathbf{P}(T) = \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{3}{5} = \frac{12}{125}.$$

447 Example An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn in succession from the urn *with replacement*, and their colour noted. What is the probability that a red, a white and another white marble will be drawn, in this order?

Solution: Since the marbles are replaced, the probability of successive drawings is not affected by previous drawings. The probability sought is thus

$$\frac{4}{12} \cdot \frac{3}{12} \cdot \frac{3}{12} = \frac{1}{48}.$$

448 Example Two numbers X and Y are chosen at random, and with replacement, from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the probability that $X^2 - Y^2$ be divisible by 3.

Solution: Notice that the sample space of this experiment has size $10 \cdot 10$ since X and Y are chosen with replacement. Observe that if $N = 3k$ then $N^2 = 9k^2$, leaves remainder 0 upon division by 3. If $N = 3k+1$ then $N^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$ leaves remainder 1 upon division by 3. Also, if $N = 3k+2$ then $N^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$ leaves remainder 1 upon division by 3. Observe that there are 3 numbers—$3, 6, 9$—divisible by 3 in the set, 4 numbers—$1, 4, 7, 10$—of the form $3k + 1$, and 3 numbers—$2, 5, 8$—of the form $3k + 2$ in the set. Now, $X^2 - Y^2$ is divisible by 3 in the following cases: (i) both X and Y are divisible by 3, (ii) both X and Y are of the form $3k + 1$, (iii) both X and Y are of the form $3k + 2$, (iv) X is of the form $3k + 1$ and Y of the form $3k+2$, (v) X is of the form $3k+2$ and Y of the form $3k+1$. Case (i) occurs $3 \cdot 3 = 9$ instances, case (ii) occurs in $4 \cdot 4 = 16$ instances, case (iii) occurs in $3 \cdot 3 = 9$ instances, case (iv) occurs in $4 \cdot 3 = 12$ instances and case (v) occurs in $3 \cdot 4 = 12$ instances. The favourable cases are thus $9 + 16 + 9 + 12 + 12 = 58$ in number and the desired probability is $\frac{58}{100} = \frac{29}{50}$.

449 Example A book has 4 typos. After each re-reading, an uncorrected typo is corrected with probability $\frac{1}{3}$. The correction of different typos is each independent one from the other. Each of the re-readings is also independent one from the other. How many re-readings are necessary so that the probability that there be no more errors be greater than 0.9?

Solution: Suppose there are n re-reading necessary in order that there be no errors. At each re-reading, the probability that a typo is not corrected is $\frac{2}{3}$. Thus the probability that a particular typo is never corrected is $(\frac{2}{3})^n$. Hence the probability that a particular typo is corrected in the n re-readings is $1-(\frac{2}{3})^n$. Thus the probability that all typos are corrected is

$$\left(1-\left(\frac{2}{3}\right)^n\right)^4.$$

We need

$$\left(1-\left(\frac{2}{3}\right)^n\right)^4 \geq 0.9$$

and with a calculator we may verify that this happens for $n \geq 10$.

Homework

450 Problem Suppose that a monkey is seated at a computer keyboard and randomly strikes the 26 letter keys and the space bar. Find the probability that its first 48 characters typed (including spaces) will be: **"the slithy toves did gyre and gimble in the wabe"**[2].

Answer: $\left(\frac{1}{27}\right)^{48}$

451 Problem An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn in succession from the urn *with*

[2] From Lewis Carroll's *The Jabberwock.*

replacement, and their colour noted. What is the probability that a red, a white and a blue marble will be drawn, in this order?

Answer: $\frac{5}{144}$

452 Problem A fair coin is tossed three times in succession. What is the probability of obtaining exactly two heads?

Answer: $\frac{3}{8}$

453 Problem Two cards are drawn in succession and with replacement from an ordinary deck of cards. What is the probability that the first card is a heart and the second one a queen?

Answer: $\frac{1}{52}$

454 Problem There are 20 tickets numbered **0**, 40 numbered **1** and and 40 numbered **2** in a hat. Two tickets **X** and **Y** are drawn at random, with replacement. Determine $\mathbf{P}(|\mathbf{X}-\mathbf{Y}|=1)$.

Answer: 0.48

455 Problem Two numbers X and Y are chosen at random, and with replacement, from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the probability that $X^2 - Y^2$ be divisible by 2.

Answer: $\frac{41}{81}$

456 Problem Events A and B are independent, events A and C are mutually exclusive, and events B and C are independent. If $\mathbf{P}(A) = \frac{1}{2}$, $\mathbf{P}(B) = \frac{1}{4}$, $\mathbf{P}(C) = \frac{1}{8}$, find $\mathbf{P}(A \cup B \cup C)$.

Hint: Theorem 373. Answer: $\frac{23}{32}$

457 Problem A die is loaded so that if **D** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{D}=k)=\frac{k}{21}$, where $k=1,2,3,4,5,6$. Another die is loaded differently, so that if **X** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{X}=k)=\frac{k^2}{91}$. Complete the following table for the distribution of $\mathbf{D}+\mathbf{X}$.

$\mathbf{D}+\mathbf{X}$	$\mathbf{P}(\mathbf{D}+\mathbf{X})$
2	$\frac{1}{91}\cdot\frac{1}{21}=\frac{1}{1911}$
3	
4	$\frac{1}{91}\cdot\frac{3}{21}+\frac{4}{91}\cdot\frac{2}{21}+\frac{9}{91}\cdot\frac{1}{21}=\frac{20}{1911}$
5	
6	
7	$\frac{1}{91}\cdot\frac{6}{21}+\frac{4}{91}\cdot\frac{5}{21}+\frac{9}{91}\cdot\frac{4}{21}+\frac{16}{91}\cdot\frac{3}{21}+\frac{25}{91}\cdot\frac{2}{21}+\frac{36}{91}\cdot\frac{1}{21}=\frac{4}{39}$
8	
9	
10	
11	$\frac{36}{91}\cdot\frac{5}{21}+\frac{25}{36}\cdot\frac{6}{21}=\frac{110}{637}$
12	$\frac{36}{91}\cdot\frac{6}{21}=\frac{72}{637}$

3.4 Binomial Random Variables

458 Definition A random variable $\mathbf{X}$ has a binomial probability distribution if

$$\mathbf{P}(\mathbf{X}=k)=\binom{n}{k}p^k(1-p)^{n-k}, \quad k=0,1,\ldots,n.$$

where n is the number of trials, p is the probability of success in one trial, and k is the number of successes.

Since

$$\sum_{k=0}^{n}\mathbf{P}(\mathbf{X}=k)=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}=(p+(1-p))^n=1,$$

this is a bonafide random variable.

459 Example A fair coin is tossed 5 times.

❶ Find the probability of obtaining 3 heads.

❷ Find the probability of obtaining 3 tails.

❸ Find the probability of obtaining at most one head.

Solution:

❶ Let $\mathbf{X}$ be the random variables counting the number of heads. Here $p=1-p=\frac{1}{2}$. Hence

$$\mathbf{P}(\mathbf{X}=3)=\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{5}{16}.$$

❷ Obtaining 3 tails is equivalent to obtaining 2 heads, hence the probability sought is

$$\mathbf{P}(\mathbf{X}=2)=\binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3=\frac{5}{16}.$$

❸ This is the probability of obtaining no heads or one head:

$$\begin{aligned}\mathbf{P}(\mathbf{X}=0)+\mathbf{P}(\mathbf{X}=1) &= \binom{5}{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{5}+\binom{5}{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{4}\\ &= \frac{1}{32}+\frac{5}{32}\\ &= \frac{3}{16}.\end{aligned}$$

460 Example A multiple-choice exam consists of 10 questions, and each question has 3 choices. It is assumed that for every question one, and only one of the choices is the correct answer.

❶ Find n, the number of trials, p, the probability of success, and $1-p$, the probability of failure.

❷ Find the probability of answering exactly 7 questions right.

❸ Find the probability of answering 8 or more questions right.

❹ Find the probability of answering at most one question.

Solution:

❶ Clearly $n=10$, $p=\frac{1}{4}$, and also, $1-p=\frac{3}{4}$.

❷ Let **X** be the random variables counting the number of right questions. Then

$$\mathbf{P}(\mathbf{X}=7)=\binom{10}{7}\left(\frac{1}{4}\right)^{7}\left(\frac{3}{4}\right)^{3}=\frac{405}{131072}.$$

❸ This is the probability of answering 8 or 9 or 10 questions right, so it is

$$\begin{aligned}
\mathbf{P}(\mathbf{X}=8)+\mathbf{P}(\mathbf{X}=9)+\mathbf{P}(\mathbf{X}=10) &= \binom{10}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2 \\
&\quad +\binom{10}{9}\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^1 \\
&\quad +\binom{10}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^0 \\
&= \frac{405}{1048576}+\frac{15}{524288}+\frac{1}{1048576} \\
&= \frac{109}{262144}.
\end{aligned}$$

461 Example A coin is loaded so that $\mathbf{P}(H) = \frac{3}{4}$ and $\mathbf{P}(T) = \frac{1}{4}$. The coin is flipped 5 times and its outcome recorded. Find the probability that heads turns up at least once.

Solution: Let A denote the event whose probability we seek. Then $\complement A$ is the event that no heads turns up. Thus

$$\mathbf{P}(\complement A) = \binom{5}{5}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 = \frac{1}{1024}.$$

Hence

$$\mathbf{P}(A) = 1 - \mathbf{P}(\complement A) = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Notice that if we wanted to find this probability directly, we would have to add the five terms

$$\begin{aligned}
\mathbf{P}(A) &= \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4+\binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3+\binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 \\
&\quad +\binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1+\binom{5}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0. \\
&= \frac{15}{1024}+\frac{90}{1024}+\frac{270}{1024}+\frac{405}{1024}+\frac{243}{1024} \\
&= \frac{1023}{1024}.
\end{aligned}$$

Homework

462 Problem A factory produces bolts, and it is known that with probability $1 - p$, a bolt is defective. Bolts are placed in containers of n bolts each. Let **D** be the random variable counting how many defective bolts there are in a container.

❶ Find $\mathbf{P}(\mathbf{D} = 2)$.

❷ Find $\mathbf{P}(\mathbf{D} \leq 2)$.

❸ Find $\mathbf{P}(\mathbf{D} \geq n - 1)$.

463 Problem When two fair coins are tossed, what is the probability of getting no heads exactly four times in five tosses?

Answer: $\frac{15}{1024}$

464 Problem A fair coin is to be flipped 1000 times. What is the probability that the number of heads exceeds the number of tails?

Answer: $\frac{1}{2} - \frac{\binom{1000}{500}}{2^{1001}}$

465 Problem In a certain game John's skill is to Peter's as 3 to 2. Find the chance of John winning 3 games at least out of 5.

Answer: $\frac{2133}{3125}$

466 Problem A coin whose faces are marked 2 and 3 is thrown 5 times. What is the chance of obtaining a total of 12?

Answer: $\frac{5}{16}$

3.5 Geometric Random Variables

467 Definition Let $0 < p < 1$. A random variable is said to have a *geometric or Pascal* distribution if

$$\mathbf{P}(\mathbf{X} = k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \ldots.$$

Thus the random variable **X** counts the number of trials necessary until success occurs.

Since

$$\sum_{k=1}^{\infty} \mathbf{P}(\mathbf{X} = k) = \sum_{k=1}^{\infty} (1-p)^{k-1}p = \frac{p}{1-(1-p)} = 1,$$

this is a bonafide random variable.

468 Example An urn contains 5 white, 4 black, and 1 red marble. Marbles are drawn, *with replacement*, until a red one is found. If **X** is the random variable counting the number of trials until a red marble appears, then

❶ $\mathbf{P}(\mathbf{X} = 1) = \frac{1}{10}$ is the probability that the marble appears on the first trial.

❷ $\mathbf{P}(\mathbf{X} = 2) = \frac{9}{10} \cdot \frac{1}{10} = \frac{9}{100}$ is the probability that the red marble appears on the second trial.

❸ $\mathbf{P}(\mathbf{X} = k) = \frac{9^{k-1}}{10^k}$ is the probability that the marble appears on the k-th trial.

469 Example A drunk has five keys in his key-chain, and an only one will start the car (caution: don't drink and drive!). He tries each key until he finds the right one (he is so drunk that he may repeat the wrong key several times), then he starts his car and (by cheer luck), arrives home safely, where his wife is waiting for him, frying pan in hand. If **X** is the random variable counting the number of trials until he find the right key, then

❶ $\mathbf{P}(\mathbf{X}=1)=\frac{1}{5}$ is the probability that he finds the key on the first trial.

❷ $\mathbf{P}(\mathbf{X}=2)=\frac{4}{5}\cdot\frac{1}{5}=\frac{4}{25}$ is the probability that he finds the key on the second trial.

❸ $\mathbf{P}(\mathbf{X}=3)=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{16}{125}$ is the probability that he finds the key on the third trial.

❹ $\mathbf{P}(\mathbf{X}=4)=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{64}{625}$ is the probability that he finds the key on the fourth trial.

❺ $\mathbf{P}(\mathbf{X}=5)=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{256}{3125}$ is the probability that he finds the key on the fifth trial.

❻ $\mathbf{P}(\mathbf{X}=6)=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{1024}{15625}$ is the probability that he finds the key on the sixth trial.

470 Example An urn contains 5 white, 4 black, and 1 red marble. Marbles are drawn, *with replacement*, until a red one is found. If **X** is the random variable counting the number of trials until a red marble appears.

❶ Find the probability that it takes at most 3 trials to obtain a red marble.

❷ Find the probability that it takes more than 3 trials to obtain a red marble.

Solution:

❶ This is asking for $\mathbf{P}(\mathbf{X}=1)+\mathbf{P}(\mathbf{X}=2)+\mathbf{P}(\mathbf{X}=3)=\frac{1}{10}+\frac{9}{100}+\frac{81}{1000}=\frac{271}{1000}$.

❷ This is asking for the infinite geometric sum

$$\mathbf{P}(\mathbf{X} > 3) = \sum_{k=4}^{\infty} \mathbf{P}(\mathbf{X} = k) = \sum_{k=4}^{\infty} \frac{9^{k-1}}{10^k}.$$

We can sum this directly, or we may resort to the fact that the event "more than 3 trials" is complementary to the event "at most 3 trials." Thus

$$\mathbf{P}(\mathbf{X} > 3) = 1 - (\mathbf{P}(\mathbf{X} = 1) + \mathbf{P}(\mathbf{X} = 2) + \mathbf{P}(\mathbf{X} = 3)) = 1 - \frac{271}{1000} = \frac{729}{1000}.$$

471 Example Three people, X, Y, Z, in order, roll a fair die. The first one to roll an even number wins and the game is ended. What is the probability that X will win?

Solution: We have

$$\begin{aligned}\mathbf{P}(\text{X wins}) &= \mathbf{P}(\text{X wins on the first trial}) \\ &\quad + \mathbf{P}(\text{X wins on the fourth trial}) \\ &\quad + \mathbf{P}(\text{X wins on the seventh trial}) + \cdots \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^6 + \cdots \\ &= \frac{\frac{1}{2}}{1 - \frac{1}{2^3}} \\ &= \frac{4}{7}.\end{aligned}$$

Homework

472 Problem Two people, X, Y, in order, roll a die. The first one to roll either a 3 or a 6 wins and the game is ended.

❶ What is the probability of throwing either a 3 or a 6?

❷ What is the probability that Y will win on the second throw?

❸ What is the probability that Y will win on the fourth throw?

❹ What is the probability that Y will win?

Answer: $\frac{1}{3}$; $\frac{2}{9}$; $\frac{8}{81}$; $\frac{2}{5}$

473 Problem Six persons throw for a stake, which is to be won by the one who first throws head with a penny; if they throw in succession, find the chance of the fourth person.

Answer: $\frac{4}{63}$

3.6 Expectation and Variance

474 Definition Let **X** be a discrete random variable taking on the values $x_1, x_2, \ldots, x_k, \ldots$. The *mean value or expectation of* **X**, denoted by $\mathbf{E}(\mathbf{X})$ is defined by

$$\mathbf{E}(\mathbf{X}) = \sum_{k=1}^{\infty} x_k \mathbf{P}(\mathbf{X} = x_k).$$

475 Example A player is paid \$1 for getting heads when flipping a fair coin and he loses \$0.50 if he gets tails.

❶ Let **G** denote the random variables measuring his gain. What is the image of **G**?

❷ Find the distribution of **G**.

❸ What is his expected gain in the long run?

Solution:

❶ **G** can either be 1 or 0.50.

❷ $\mathbf{P}(\mathbf{G} = 1) = \frac{1}{2}$, and $\mathbf{P}(\mathbf{G} = 0.5) = \frac{1}{2}$,

❸

$$\mathbf{E}(\mathbf{G}) = 1\mathbf{P}(\mathbf{G} = 1) + 0.5\mathbf{P}(\mathbf{G} = 0.5) = \frac{3}{4}.$$

476 Example A player is playing with a fair die. He gets \$2 if the die lands on a prime, he gets nothing if the die lands on 1, and he loses \$1 if the die lands on a composite number.

❶ Let **G** denote the random variables measuring his gain. What is the image of **G**?

❷ Find the distribution of **G**.

❸ What is his expected gain in the long run?

Solution:

❶ **G** can either be 2, 0 or -1.

❷ $\mathbf{P}(\mathbf{G}=2)=\frac{3}{6}$, $\mathbf{P}(\mathbf{G}=0)=\frac{1}{6}$, and $\mathbf{P}(\mathbf{G}=-1)=\frac{2}{6}$.

❸
$$\mathbf{E}(\mathbf{G})=2\mathbf{P}(\mathbf{G}=2)+0\mathbf{P}(\mathbf{G}=0)-1\mathbf{P}(\mathbf{G}=-1)=\frac{6}{6}+0-\frac{2}{6}=\frac{2}{3}.$$

477 Example A player chooses, without replacement, two cards from a standard deck of cards. He gets \$2 for each heart suit card.

❶ Let **G** denote the random variables measuring his gain. What is the image of **G**?

❷ Find the distribution of **G**.

❸ What is his expected gain in the long run?

Solution:

❶ **G** can either be 0, 1 or 2.

❷
$$\mathbf{P}(\mathbf{G}=0)=\frac{\binom{13}{0}\binom{39}{2}}{\binom{52}{2}}=\frac{19}{34},$$
$$\mathbf{P}(\mathbf{G}=1)=\frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}}=\frac{13}{34},$$
and
$$\mathbf{P}(\mathbf{G}=2)=\frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}}=\frac{1}{17}.$$

❸
$$\mathbf{E}(\mathbf{G})=0\mathbf{P}(\mathbf{G}=0)+1\mathbf{P}(\mathbf{G}=1)+2\mathbf{P}(\mathbf{G}=2)=0+\frac{13}{34}+\frac{2}{17}=\frac{1}{2}.$$

478 Definition Let $\mathbf{X}$ be a discrete random variable taking on the values $x_1, x_2, \ldots, x_k, \ldots$. Then $\mathbf{E}\left(\mathbf{X}^2\right)$ is defined by

$$\mathbf{E}\left(\mathbf{X}^2\right) = \sum_{k=1}^{\infty} x_k^2 \mathbf{P}\left(\mathbf{X} = x_k\right).$$

479 Definition Let $\mathbf{X}$ be a random variable. The *variance* $\text{var}\left(\mathbf{X}\right)$ of $\mathbf{X}$ is defined by

$$\text{var}\left(\mathbf{X}\right) = \mathbf{E}\left(\mathbf{X}^2\right) - \left(\mathbf{E}\left(\mathbf{X}\right)\right)^2.$$

480 Example A random variable has distribution function as shewn below.

$\mathbf{X}$	$\mathbf{P}\left(\mathbf{X}\right)$
-1	$2k$
1	$3k$
2	$4k$

❶ Find the value of k.

❷ Determine the actual values of $\mathbf{P}\left(\mathbf{X} = -1\right)$, $\mathbf{P}\left(\mathbf{X} = 1\right)$, and $\mathbf{P}\left(\mathbf{X} = 2\right)$.

❸ Find $\mathbf{E}\left(\mathbf{X}\right)$.

❹ Find $\mathbf{E}\left(\mathbf{X}^2\right)$.

❺ Find $\text{var}\left(\mathbf{X}\right)$.

Solution:

❶ The probabilities must add up to 1:

$$2k + 3k + 4k = 1 \implies k = \frac{1}{9}.$$

❷

$$\mathbf{P}(\mathbf{X} = -1) = 2k = \frac{2}{9},$$

$$\mathbf{P}(\mathbf{X} = 1) = 3k = \frac{3}{9},$$

$$\mathbf{P}(\mathbf{X} = 2) = 4k = \frac{4}{9}.$$

❸

$$\mathbf{E}(\mathbf{X}) = -1\mathbf{P}(\mathbf{X} = -1) + 1\mathbf{P}(\mathbf{X} = 1) + 2\mathbf{P}(\mathbf{X} = 2) = -1\cdot\frac{2}{9} + 1\cdot\frac{3}{9} + 2\cdot\frac{4}{9} = 1.$$

❹

$$\mathbf{E}\left(\mathbf{X}^2\right) = (-1)^2\mathbf{P}(\mathbf{X} = -1) + 1^2\mathbf{P}(\mathbf{X} = 1) + 2^2\mathbf{P}(\mathbf{X} = 2) = 1\cdot\frac{2}{9} + 1\cdot\frac{3}{9} + 4\cdot\frac{4}{9} = \frac{21}{9}.$$

❺

$$\mathrm{var}(\mathbf{X}) = \mathbf{E}\left(\mathbf{X}^2\right) - (\mathbf{E}(\mathbf{X}))^2 = \frac{21}{9} - 1^2 = \frac{4}{3}.$$

481 Example John and Peter play the following game with three fair coins: John plays a stake of \$10 and tosses the three coins in turn. If he obtains three heads, his stake is returned together with a prize of \$30. For two consecutive heads, his stake money is returned, together with a prize of \$10. In all other cases, Peter wins the stake money. Is the game fair?

Solution: The game is fair if the expected gain of both players is the same. Let **J** be the random variable measuring John's gain and let **P** be the random variable measuring Peter's gain. John wins when the coins shew HHH, HHT, THH. Thus

$$\begin{aligned}\mathbf{E}(\mathbf{J}) &= 30\mathbf{P}(HHH) + 10\mathbf{P}(HHT) + 10\mathbf{P}(THH)\\ &= 30\cdot\frac{1}{8} + 10\cdot\frac{1}{8} + 10\cdot\frac{1}{8}\\ &= \frac{25}{4}.\end{aligned}$$

Peter wins when the coins shew HTH, HTT, THT, TTH, TTT. Thus

$$\begin{aligned}\mathbf{E}(\mathbf{P}) &= 10\mathbf{P}(\text{HTH}) + 10\mathbf{P}(\text{HTT}) + 10\mathbf{P}(\text{THT}) + 10\mathbf{P}(\text{TTH}) + 10\mathbf{P}(\text{TTT}) \\ &= 10\cdot\frac{1}{8} + 10\cdot\frac{1}{8} + 10\cdot\frac{1}{8} + 10\cdot\frac{1}{8} + 10\cdot\frac{1}{8} \\ &= \frac{25}{4},\end{aligned}$$

whence the game is fair.

Homework

482 Problem A fair die is tossed. If the resulting number is even, you multiply your score by 2 and get that many dollars. If the resulting number is odd, you add 1 to your score and get that many dollars. Let **X** be the random variable counting your gain, in dollars.

❶ Give the range of **X**.

❷ Give the distribution of **X**.

❸ Find $\mathbf{E}(\mathbf{X})$.

❹ Find $\text{var}(\mathbf{X})$.

483 Problem Consider the random variable **X** with distribution table as follows.

X	$\mathbf{P}(\mathbf{X})$
-2	0.3
-1	k
0	$5k$
1	$2k$

❶ Find the value of k.

❷ Find $\mathbf{E}(\mathbf{X})$.

❸ Find $\mathbf{E}\left(\mathbf{X}^2\right)$.

❹ Find $\operatorname{var}(\mathbf{X})$.

Answer: 0.0875; -0.5125 ; 1.4625 ; 1.19984375

484 Problem A fair coin is to be tossed thrice. The player receives $\$10$ if all three tosses turn up heads, and pays $\$3$ if there is one or no heads. No gain or loss is incurred otherwise. If **Y** is the gain of the player, find $E\mathbf{Y}$.

Answer: -0.25

485 Problem A die is loaded so that if **D** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{D}=k) = \frac{k}{21}$, where $k = 1, 2, 3, 4, 5, 6$. Another die is loaded differently, so that if **X** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{X}=k) = \frac{k^2}{91}$.

❶ Find the expectation $\mathbf{E}(\mathbf{D}+\mathbf{X})$.

❷ Find the variance $\operatorname{var}(\mathbf{D}+\mathbf{X})$.

486 Problem John and Peter each put \$1 into a pot. They then decide to throw a pair of dice alternately (John plays first, Peter second, then John again, etc.). The first one who throws a 5 wins the pot. How much money should John add to the pot in order to make the game fair?

Answer: $\$\frac{1}{8}$

487 Problem A man pays \$1 to throw three fair dice. If at least one 6 appears, he receives back his stake together with a prize consisting of the number of dollars equal to the number of sixes shewn. Does he expect to win or lose?

Answer: Lose

Chapter 4

Conditional Probability

4.1 Conditional Probability

488 Definition Given an event B, the probability that event A happens given that event B has occurred is defined and denoted by

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \quad \mathbf{P}(B) \neq 0.$$

489 Example Ten cards numbered 1 through 10 are placed in a hat, mixed and then one card is pulled at random. If the card is an even numbered card, what is the probability that its number is divisible by 3?

Solution: Let A be the event "the card's number is divisible by 3" and B be the event "the card is an even numbered card." We want $\mathbf{P}(A|B)$. Observe that $\mathbf{P}(B) = \frac{5}{10} = \frac{1}{2}$. Now the event $A \cap B$ is the event that the card's number is both even and divisible by 3, which happens only when the number of the card is 6. Hence $\mathbf{P}(A \cap B) = \frac{1}{10}$. The desired probability is

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\frac{1}{10}}{\frac{1}{2}} = \frac{1}{5}.$$

490 Example A coin is tossed twice. What is the probability that in both tosses appear heads given that in at least one of the tosses appeared heads?

Solution: Let $E = \{(H, H)\}$ and $F = \{(H, H), (H, T), (T, H)\}$. Then

$$\mathbf{P}(E|F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(\{(H, H)\})}{\mathbf{P}(\{(H, H), (H, T), (T, H)\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

The conditional probability formula can be used to obtain probabilities of intersections of events. Thus

$$\mathbf{P}(A \cap B) = \mathbf{P}(B)\,\mathbf{P}(A|B) \tag{4.1}$$

Observe that the sinistral side of the above equation is symmetric. Thus we similarly have

$$\mathbf{P}(A \cap B) = \mathbf{P}(B \cap A) = \mathbf{P}(A)\,\mathbf{P}(B|A) \tag{4.2}$$

491 Example Darlene is undecided on whether taking Statistics or Philosophy. She knows that if she takes Statistics she will get an A with probability $\frac{1}{3}$, while if she takes Philosophy she will receive an A with probability $\frac{1}{2}$. Darlene bases her decision on the flip of a coin. What is the probability that Darlene will receive an A in Statistics?

Solution: Let E be the event that Darlene takes Statistics and let F be the event that she receives an A in whatever course she decides to take. Then we want $\mathbf{P}(E \cap F)$. But

$$\mathbf{P}(E \cap F) = \mathbf{P}(E)\,\mathbf{P}(F|E) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

492 Example An urn contains eight black balls and three white balls. We draw two balls without replacement. What is the probability that both balls are black?

Solution: Let B_1 be the event that the first ball is black and let B_2 be the event that the second ball is black. Clearly $\mathbf{P}(B_1) = \frac{8}{11}$. If a black ball is taken out, there remain 10 balls in the urn, 7 of which are black. Thus $\mathbf{P}(B_2|B_1) = \frac{7}{10}$. We conclude that

$$\mathbf{P}(B_1 \cap B_2) = \mathbf{P}(B_1)\,\mathbf{P}(B_2|B_1) = \frac{8}{11} \cdot \frac{7}{10} = \frac{28}{55}.$$

The formula for conditional probability can be generalised to any number of events. Thus if $A_1, A_2, \ldots A_n$ are events, then

$$\begin{aligned}\mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_n) &= \mathbf{P}(A_1) \\ &\cdot \mathbf{P}(A_2|A_1)\,\mathbf{P}(A_3|A_1 \cap A_2) \\ &\cdots \mathbf{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})\end{aligned} \tag{4.3}$$

493 Example An urn contains 5 red marbles, 4 blue marbles, and 3 white marbles. Three marbles are drawn in succession, without replacement. Find the probability that the first two are white and the third one is blue.

Solution: Let the required events be W_1, W_2, B_3. Then

$$\mathbf{P}(W_1 \cap W_2 \cap B_3) = \mathbf{P}(W_1)\,\mathbf{P}(W_2|W_1)\,\mathbf{P}(B_3|W_1 \cap W_2) = \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{4}{10} = \frac{1}{55}.$$

Homework

494 Problem Two cards are drawn in succession from a well-shuffled standard deck of cards. What is the probability of successively obtaining

❶ a red card and then a black card?

❷ two red cards?

❸ a knave and then a queen?

❹ two knaves?

Answer: $\frac{13}{51}$; $\frac{25}{102}$; $\frac{4}{663}$; $\frac{1}{221}$

495 Problem Five cards are drawn at random from a standard deck of cards. It is noticed that there is at least one picture (A, J, Q, or K) card. Find the probability that this hand of cards has two knaves.

Answer: $\frac{1}{116}$

496 Problem Five cards are drawn at random from a standard deck of cards. It is noticed that there is exactly one ace card. Find the probability that this hand of cards has two knaves.

Answer: $\frac{473}{16215}$

4.2 Conditioning

Sometimes we may use the technique of *conditioning*, which consists in decomposing an event into mutually exclusive parts. Let E and F be events. Then

$$\begin{aligned} \mathbf{P}(E) &= \mathbf{P}(E \cap F) + \mathbf{P}\left(E \cap \complement F\right) \\ &= \mathbf{P}(F)\,\mathbf{P}(E|F) + \mathbf{P}\left(\complement F\right)\mathbf{P}\left(E|\complement F\right). \end{aligned} \tag{4.4}$$

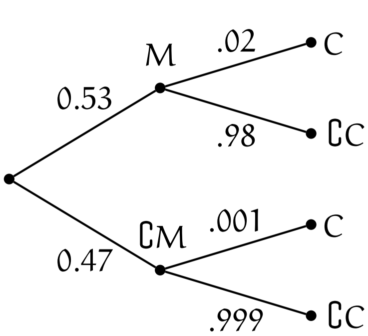


Figure 4.1: Example 497.

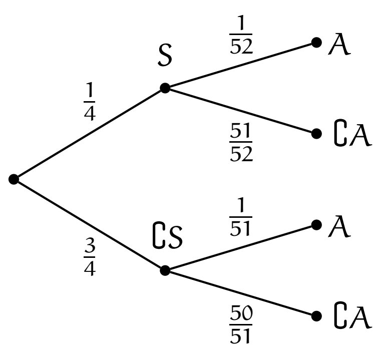


Figure 4.2: Example 498.

497 Example A population consists of 53% men. The probability of colour blindness is .02 for a man and .001 for a woman. Find the probability that a person picked at random is colour blind.

Solution: We condition on the sex of the person. Let M be the event that the person is a man and let C be the event that the person is colour-blind. Then

$$\mathbf{P}(C) = \mathbf{P}(C \cap M) + \mathbf{P}\left(C \cap \complement M\right).$$

But $\mathbf{P}(C \cap M) = \mathbf{P}(M)\mathbf{P}(C|M) = (.53)(.02) = 0.106$ and $\mathbf{P}\left(C \cap \complement M\right) = \mathbf{P}\left(\complement M\right)\mathbf{P}\left(C|\complement M\right) = (.47)(.001) = .00047$ and so $\mathbf{P}(C) = 0.10647$. A tree diagram explaining this calculation can be seen in figure 4.1.

498 Example Draw a card. If it is a spade, put it back and draw a second card. If the first card is not a spade, draw a second card without replacing the second one. Find the probability that the second card is the ace of spades.

Solution: We condition on the first card. Let S be the event that the first card is a spade and let A be the event that the second card is the ace of spades. Then

$$\mathbf{P}(A) = \mathbf{P}(A \cap S) + \mathbf{P}\left(A \cap \complement S\right).$$

But $\mathbf{P}(A \cap S) = \mathbf{P}(S)\mathbf{P}(A|S) = \frac{1}{4}\cdot\frac{1}{52} = \frac{1}{108}$ and $\mathbf{P}\left(A \cap \complement S\right) = \mathbf{P}\left(\complement S\right)\mathbf{P}\left(A|\complement S\right) = \frac{3}{4}\cdot\frac{1}{51} = \frac{1}{68}$. We thus have

$$\mathbf{P}(A) = \frac{1}{108} + \frac{1}{68} = \frac{11}{459}.$$

A tree diagram explaining this calculation can be seen in figure 4.2.

499 Example A multiple-choice test consists of five choices per question. You think you know the answer for 75% of the questions and for the other 25% you guess at random. When you think you know the answer, you are right only 80% of the time. Find the probability of getting an arbitrary question right.

Solution: We condition on whether you think you know the answer to the question. Let K be the event that you think you know the answer to the question and let R be the event that you get a question right. Then

$$\mathbf{P}(R) = \mathbf{P}(K \cap R) + \mathbf{P}\left(\complement K \cap R\right)$$

Now $\mathbf{P}(K \cap R) = \mathbf{P}(K)\,\mathbf{P}(R|K) = (.75)(.8) = .6$ and

$$\mathbf{P}\left(\complement K \cap R\right) = \mathbf{P}\left(\complement K\right)\mathbf{P}\left(R|\complement K\right) = (.25)(.2) = .05.$$

Therefore $\mathbf{P}(R) = .6 + .05 = .65$.

If instead of conditioning on two disjoint sets we conditioned in n pairwise disjoint sets, we would obtain

500 Theorem (Law of Total Probability) Let $F = F_1 \cup F_2 \cup \cdots \cup F_n$, where $F_j \cap F_k = \varnothing$ if $j \neq k$, then

$$\mathbf{P}(E \cap F) = \mathbf{P}(F_1)\,\mathbf{P}(E|F_1) + \mathbf{P}(F_2)\,\mathbf{P}(E|F_2) + \cdots + \mathbf{P}(F_n)\,\mathbf{P}(E|F_n).$$

501 Example An urn contains 4 red marbles and 5 green marbles. A marble is selected at random and its colour noted, then this marble is put back into the urn. If it is red, then 2 more red marbles are put into the urn and if it is green 1 more green marble is put into the urn. A second marble is taken from the urn. Let R_1, R_2 be the events that we select a red marble on the first and second trials respectively, and let G_1, G_2 be the events that we select a green marble on the first and second trials respectively.

❶ Find $\mathbf{P}(R_2)$.

❷ Find $\mathbf{P}(R_2 \cap R_1)$.

❸ Find $\mathbf{P}(R_1|R_2)$.

Solution:

❶

$$\mathbf{P}(R_2) = \frac{4}{9}\cdot\frac{6}{11} + \frac{5}{9}\cdot\frac{2}{5} = \frac{46}{99}.$$

❷

$$\mathbf{P}(R_2 \cap R_1) = \frac{4}{9}\cdot\frac{6}{11} = \frac{8}{33}$$

❸

$$\mathbf{P}(R_1|R_2) = \frac{\mathbf{P}(R_2 \cap R_1)}{\mathbf{P}(R_2)} = \frac{24}{46} = \frac{12}{23}.$$

502 Example An urn contains 10 marbles: 4 red and 6 blue. A second urn contains 16 red marbles and an unknown number of blue marbles. A single marble is drawn from each urn. The probability that both marbles are the same color is 0.44. Calculate the number of blue marbles in the second urn.

Solution: Let b be the number of blue marbles in the second urn, let $R_k, k = 1, 2$ denote the event of drawing a red marble from urn k, and similarly define $B_k, k = 1, 2$. We want

$$\mathbf{P}\left((R_1 \cap R_2) \cup (B_1 \cap B_2)\right).$$

Observe that the events $R_1 \cap R_2$ and $B_1 \cap B_2$ are mutually exclusive, and that R_1 is independent of R_2 and B_1 is independent of B_2 (drawing a marble from the first urn does not influence drawing a second marble from the second urn). We then have

$$\begin{aligned} 0.44 &= \mathbf{P}\left((R_1 \cap R_2) \cup (B_1 \cap B_2)\right) \\ &= \mathbf{P}\left(R_1 \cap R_2\right) + \mathbf{P}\left(B_1 \cap B_2\right) \\ &= \mathbf{P}\left(R_1\right)\mathbf{P}\left(R_2\right) + \mathbf{P}\left(B_1\right)\mathbf{P}\left(B_2\right) \\ &= \frac{4}{10} \cdot \frac{16}{b+16} + \frac{6}{10} \cdot \frac{b}{b+16}. \end{aligned}$$

Clearing denominators

$$0.44(10)(b+16) = 4(16) + 6b \implies b = 4.$$

503 Example (Monty Hall Problem) You are on a television shew where the host shews you three doors. Behind two of them are goats, and behind the remaining one a car. You choose one door, but the door is not yet opened. The host opens a door that has a goat behind it (he never opens the door that hides the car), and asks you whether you would like to switch your door to the unopened door. Should you switch?

Solution: It turns out that by switching, the probability of getting the car increases from $\frac{1}{3}$ to $\frac{2}{3}$. Let us consider the following generalisation: an urn contains a white marbles and b black marbles with $a + b \geq 3$. You have two strategies:

❶ You may simply draw a marble at random. If it is white you win, otherwise you lose.

❷ You draw a marble at random without looking at it, and you dispose of it. The host removes a black marble from the urn. You now remove a marble from the urn. If it is white you win, otherwise you lose.

In the first strategy your probability of winning is clearly $\frac{a}{a+b}$. To compute the probability of winning on the second strategy we condition on the colour of the marble that you first drew. The probability of winning is thus

$$\frac{a}{a+b} \cdot \frac{a-1}{a+b-2} + \frac{b}{a+b} \cdot \frac{a}{a+b-2} = \frac{a}{a+b}\left(1 + \frac{1}{a+b-2}\right).$$

This is greater than the probability on the first strategy, so the second strategy is better.

Homework

504 Problem An urn contains 5 red marbles and 5 green marbles. A marble is selected at random and its colour noted, then this marble is put back into the urn. If it is red, then 2 more red marbles are put into the urn and if it is green 3 more green marbles are put into the urn. A second marble is taken from the urn. Let R_1, R_2 be the events that we select a red marble on the first and second trials respectively, and let G_1, G_2 be the events that we select a green marble on the first and second trials respectively.

1. Find $\mathbf{P}(R_1)$.

2. Find $\mathbf{P}(G_1)$.
3. Find $\mathbf{P}(R_2|R_1)$.
4. Find $\mathbf{P}(G_2|R_1)$.
5. Find $\mathbf{P}(G_2|G_1)$.
6. Find $\mathbf{P}(R_2|G_1)$.
7. Find $\mathbf{P}(R_2)$.
8. Find $\mathbf{P}(G_2)$.
9. Find $\mathbf{P}(R_2 \cap R_1)$.
10. Find $\mathbf{P}(R_1|R_2)$.
11. Find $\mathbf{P}(G_2 \cap R_1)$.
12. Find $\mathbf{P}(R_1|G_2)$.

505 Problem A cookie jar has 3 red marbles and 1 white marble. A shoebox has 1 red marble and 1 white marble. Three marbles are chosen at random without replacement from the cookie jar and placed in the shoebox. Then 2 marbles are chosen at random and without replacement from the shoebox. What is the probability that both marbles chosen from the shoebox are red?

Answer: $\frac{3}{8}$

506 Problem A fair coin is tossed until a head appears. Given that the first head appeared on an even numbered toss, what is the conditional probability that the head appeared on the fourth toss?

Answer: $\frac{3}{16}$

507 Problem Five urns are numbered $3, 4, 5, 6$, and 7, respectively. Inside each urn is n^2 dollars where n is the number on the urn. You select an urn at random. If it is a prime number, you receive the amount in the urn. If the number is not a prime number, you select a second urn from the remaining four urns and you receive the total amount of money in the two urns selected. What is the probability that you end up with $\$25$?

Answer: $\frac{1}{4}$

508 Problem A family has five children. Assuming that the probability of a girl on each birth was $\frac{1}{2}$ and that the five births were independent, what is the probability the family has at least one girl, given that they have at least one boy?

Answer: $\frac{30}{31}$

509 Problem Events S and T have probabilities $\mathbf{P}(S) = \mathbf{P}(T) = \frac{1}{3}$ and $\mathbf{P}(S|T) = \frac{1}{6}$. What is $\mathbf{P}\left(\complement S \cap \complement T\right)$?

Answer: $\frac{7}{18}$

4.3 Bayes' Rule

Suppose $\Omega = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_j \cap A_k = \varnothing$ if $j \neq k$ is a partition of the sample space. Then

$$\mathbf{P}(A_k|B) = \frac{\mathbf{P}(A_k \cap B)}{\mathbf{P}(B)}.$$

By the Law of Total Probability Theorem 500, $\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B|A_1) + \mathbf{P}(A_2)\mathbf{P}(B|A_2) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B|A_n)$. This gives

510 Theorem (Bayes' Rule) . Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint with union Ω. Then

$$\mathbf{P}(A_k|B) = \frac{\mathbf{P}(A_k \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A_k \cap B)}{\sum_{k=1}^{n} \mathbf{P}(A_k)\mathbf{P}(B|A_k)}.$$

511 Example There are three urns, A, B, and C. Urn A has a red marbles and b green marbles, urn B has c red marbles and d green marbles, and urn C has a red marbles and c green marbles. Let A be the event of choosing urn A, B of choosing urn B and, C of choosing urn C. Let R be the event of choosing a red marble and G be the event of choosing a green marble. An urn is chosen at random, and after that, from this urn, a marble is chosen at random.

❶ Find $\mathbf{P}(G)$.

❷ Find $\mathbf{P}(G|C)$.

❸ Find $\mathbf{P}(C|G)$.

❹ Find $\mathbf{P}(R)$.

❺ Find $\mathbf{P}(R|A)$.

❻ Find $\mathbf{P}(A|R)$.

Solution:

❶ Conditioning on the urn chosen,

$$\begin{aligned} \mathbf{P}(G) &= \mathbf{P}(G|A)\,\mathbf{P}(A) + \mathbf{P}(G|B)\,\mathbf{P}(B) + \mathbf{P}(G|C)\,\mathbf{P}(C) \\ &= \frac{b}{a+b}\cdot\frac{1}{3} + \frac{d}{c+d}\cdot\frac{1}{3} + \frac{c}{a+c}\cdot\frac{1}{3}. \end{aligned}$$

❷ This is clearly $\dfrac{c}{a+c}$.

❸ We use Bayes' Rule

$$\begin{aligned} \mathbf{P}(C|G) &= \frac{\mathbf{P}(C\cap G)}{\mathbf{P}(G)} \\ &= \frac{\mathbf{P}(G|C)\,\mathbf{P}(C)}{\mathbf{P}(G)} \\ &= \frac{\frac{c}{a+c}\cdot\frac{1}{3}}{\frac{b}{a+b}\cdot\frac{1}{3} + \frac{d}{c+d}\cdot\frac{1}{3} + \frac{c}{a+c}\cdot\frac{1}{3}} \\ &= \frac{\frac{c}{a+c}}{\frac{b}{a+b} + \frac{d}{c+d} + \frac{c}{a+c}} \end{aligned}$$

❹ Conditioning on the urn chosen,

$$\begin{aligned} \mathbf{P}(R) &= \mathbf{P}(R|A)\,\mathbf{P}(A) + \mathbf{P}(R|B)\,\mathbf{P}(B) + \mathbf{P}(R|C)\,\mathbf{P}(C) \\ &= \frac{a}{a+b}\cdot\frac{1}{3} + \frac{c}{c+d}\cdot\frac{1}{3} + \frac{a}{a+c}\cdot\frac{1}{3}. \end{aligned}$$

❺ This is clearly $\dfrac{a}{a+b}$.

❻ We use Bayes' Rule

$$\begin{aligned}
\mathbf{P}(A|R) &= \frac{\mathbf{P}(A\cap R)}{\mathbf{P}(R)} \\
&= \frac{\mathbf{P}(R|C)\,\mathbf{P}(C)}{\mathbf{P}(R)} \\
&= \frac{\frac{a}{a+b}\cdot\frac{1}{3}}{\frac{a}{a+b}\cdot\frac{1}{3}+\frac{c}{c+d}\cdot\frac{1}{3}+\frac{a}{a+c}\cdot\frac{1}{3}} \\
&= \frac{\frac{a}{a+b}}{\frac{a}{a+b}+\frac{c}{c+d}+\frac{a}{a+c}}
\end{aligned}$$

512 Example Two distinguishable dice have probabilities p, and 1 respectively of throwing a 6. One of the dice is chosen at random and thrown. A 6 appeared.

❶ Find the probability of throwing a 6.

❷ What is the probability that one simultaneously chooses die I and one throws a 6?

❸ What is the probability that the die chosen was the first one?

Solution:

❶
$$\mathbf{P}(6) = \mathbf{P}(6\cap \mathrm{I}) + \mathbf{P}(6\cap \mathrm{II}) = \frac{1}{2}\cdot p + \frac{1}{2}\cdot 1 = \frac{p+1}{2}$$

❷ $\mathbf{P}(6\cap \mathrm{I}) = \frac{1}{2}\cdot p = \frac{p}{2}$

❸
$$\mathbf{P}(\mathrm{I}|6) = \frac{\mathbf{P}(6\cap \mathrm{I})}{\mathbf{P}(6)} = \frac{p}{p+1}.$$

513 Example Three boxes identical in appearance contain the following coins: Box I has two quarters and a dime; Box II has 1 quarter and 2 dimes; Box III has 1 quarter and 1 dime. A coin drawn at random from a box selected is a quarter.

❶ Find the probability of obtaining a quarter.

❷ What is the probability that one simultaneously choosing box III and getting a quarter?

❸ What is the probability that the quarter came from box III?

Solution:

❶

$$\mathbf{P}(Q) = \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}.$$

❷

$$\mathbf{P}(Q \cap III) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

❸

$$\mathbf{P}(III|Q) = \frac{\mathbf{P}(III \cap Q)}{\mathbf{P}(Q)} = \frac{1}{3}.$$

Homework

514 Problem Three dice have the following probabilities of throwing a 6: p, q, r, respectively. One of the dice is chosen at random and thrown. A 6 appeared. What is the probability that the die chosen was the first one?

Answer: $\dfrac{p}{p+q+r}$

515 Problem Three boxes identical in appearance contain the following coins: Box A has two quarters; Box B has 1 quarter and 2 dimes; Box C has 1 quarter and 1 dime. If a coin drawn at random from a box selected is a quarter, what is the probability that the randomly selected box contains at least one dime?

516 Problem An urn contains 6 red marbles and 3 green marbles. One marble is selected at random and is replaced by a marble of the other colour. A second marble is then drawn. What is the probability that the first marble selected was red given that the second one was also red?

Answer: $\frac{10}{17}$

517 Problem There are three dice. Die I is an ordinary fair die, so if **F** is the random variable giving the score on this die, then $\mathbf{P}(\mathbf{F} = k) = \frac{1}{6}$, Die II is loaded so that if **D** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{D} = k) = \frac{k}{21}$, where $k = 1, 2, 3, 4, 5, 6$. Die is loaded differently, so that if **X** is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{X} = k) = \frac{k^2}{91}$. A die is chosen at random and a 5 appears. What is the probability that it was Die II?

Answer: $\frac{91}{371}$

518 Problem There are 3 urns each containing 5 white marbles and 2 black marbles, and 2 urns each containing 1 white marble and 4 black marbles. A black marble having been drawn, find the chance that it came from the first group of urns.

$\frac{15}{43}$

519 Problem There are four marbles in an urn, but it is not known of what colours they are. One marble is drawn and found to be white. Find the probability that all the marbles are white.

Answer: $\frac{2}{5}$

520 Problem In an urn there are six marbles of unknown colours. Three marbles are drawn and found to be black. Find the chance that no black marble is left in the urn.

Answer: $\frac{1}{35}$

521 Problem John speaks the truth 3 out of 4 times. Peter speaks the truth 5 out of 6 times. What is the probability that they will contradict each other in stating the same fact?

Answer: $\frac{1}{3}$

Chapter 5

Some Continuous Random Variables

5.1 Uniform Continuous Random Variables

522 Definition Let $\mathcal{C}$ be a body in one dimension (respectively, two, or three dimensions) having positive length $\text{meas}\left(\mathcal{C}\right)$ (respectively, positive area or positive volume). A *continuous random variable* $\mathbf{X}$ *defined on* $\mathcal{C}$ is a random variable with probability given by

$$\mathbf{P}\left(\mathbf{X} \in A\right) = \frac{\text{meas}\left(\mathcal{A}\right)}{\text{meas}\left(\mathcal{C}\right)}.$$

This means that the probability of of an event is proportional to the length (respectively, area or volume) that this body $\mathcal{A}$ occupies in $\mathcal{C}$.

523 Example A dartboard is made of three concentric circles of radii 3, 5, and 7, as in figure 5.1. A dart is thrown and it is assumed that it always lands on the dartboard. Here the inner circle is blue, the middle ring is white and the outer ring is red.

❶ The size of the sample space for this experiment is $\pi(7)^2 = 49\pi$.

❷ The probability of landing on blue is $\dfrac{\pi(3)^2}{49\pi} = \dfrac{9}{49}$.

❸ The probability of landing on white is $\dfrac{\pi(5)^2 - \pi(3)^2}{49\pi} = \dfrac{16}{49}$.

❹ The probability of landing on red is $\dfrac{\pi(7)^2 - \pi(5)^2}{49\pi} = \dfrac{24}{49}$.

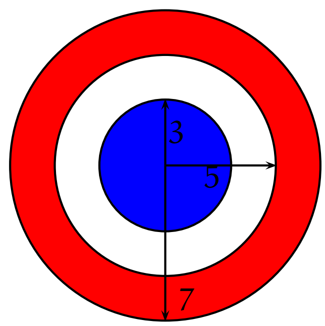


Figure 5.1: Example 523

524 Definition The *distribution function* F of a random variable $\mathbf{X}$ is $F(a) = \mathbf{P}(\mathbf{X} \leq a)$.

A distribution function satisfies

❶ If $a < b$ then $F(a) \leq F(b)$.

❷ $\lim_{a \to -\infty} F(a) = 0$,

❸ $\lim_{a \to +\infty} F(a) = 1$.

525 Example A random variable $\mathbf{X}$ has probability distribution

$$\mathbf{P}(\mathbf{X} \leq x) = \kappa \text{meas}(x),$$

where $\text{meas}(x)$ denotes the area of the polygon in figure 525 up to abscissa x. Assume that $\mathbf{P}(\mathbf{X} \leq 0) = 0$ and that $\mathbf{P}(\mathbf{X} \leq 6) = 1$.

❶ Find the value of κ.

❷ Find $\mathbf{P}(\mathbf{X} \leq 2)$.

❸ Find $\mathbf{P}(3 \leq \mathbf{X} \leq 4)$.

Solution:

❶ The figure is composed of a rectangle and a triangle, and its total area is $(4)(2) + \frac{1}{2}(4)(5) = 8 + 10 = 18$. Since $1 = \mathbf{P}(\mathbf{X} \leq 6) = \kappa \text{meas}(6) = 18\kappa$ we have $\kappa = \frac{1}{18}$.

❷ $\mathbf{P}(\mathbf{X} \leq 2)$ is the area of the rectangle between $x = 0$ and $x = 2$ and so $\mathbf{P}(\mathbf{X} \leq 2) = \frac{1}{18}(8) = \frac{4}{9}$.

❸ $\mathbf{P}(3 \leq \mathbf{X} \leq 4)$ is the area of a trapezoid of bases of length 2.5 and 5 and height 1, thus $\mathbf{P}(3 \leq \mathbf{X} \leq 4) = \frac{1}{18} \cdot \frac{1}{2}(\frac{5}{2} + 5) = \frac{5}{24}$.

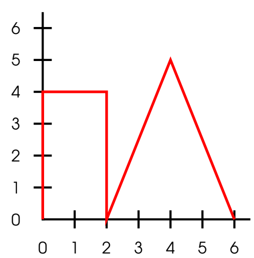


Figure 5.2: Example 525

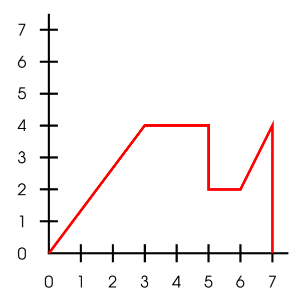


Figure 5.3: Example 526

526 Example A random variable $\mathbf{X}$ has probability distribution

$$\mathbf{P}(\mathbf{X} \leq x) = \kappa A(x),$$

where $A(x)$ denotes the area of the polygon in figure 526 up to abscissa x. Assume that $\mathbf{P}(\mathbf{X} \leq 0) = 0$ and that $\mathbf{P}(\mathbf{X} \leq 7) = 1$.

❶ Find the value of κ.

❷ Find $\mathbf{P}(\mathbf{X} \leq 3)$.

❸ Find $\mathbf{P}(\mathbf{X} \leq 5)$.

❹ Find $\mathbf{P}(\mathbf{X} \leq 6)$.

❺ Find $\mathbf{P}(1 \leq \mathbf{X} \leq 2)$.

❻ Find $\mathbf{P}(\mathbf{X} \geq 6)$.

❼ Find a median m of $\mathbf{X}$, that is, an abscissa that simultaneously satisfies $\mathbf{P}(\mathbf{X} \geq m) \geq \frac{1}{2}$ and $\mathbf{P}(\mathbf{X} \leq m) \geq \frac{1}{2}$.

Solution:

❶ In $[0;3]$ the figure is a triangle with base 3 and height 4, and so its area is 6. In $[3;5]$ the figure is a rectangle, with base 2 and height 4, and so its area is 8. In $[5;6]$ the figure is a rectangle, with base 1 and height 2, and so its area is 2. In $[6;7]$ the figure is a trapezium, with bases 2 and 4 and height 1, and so its area is 3. Adding all these areas together we obtain $6+8+2+3=19$. Since

$$1 = \mathbf{P}(\mathbf{X} \leq 7) = \kappa A(7) = \kappa(19),$$

we obtain $\kappa = \frac{1}{19}$.

❷ This measures the proportion of the area enclosed by the triangle, and so $\mathbf{P}(\mathbf{X} \leq 3) = \frac{6}{19}$.

❸ This measures the proportion of the area enclosed by the triangle and the first rectangle, and so $\mathbf{P}(\mathbf{X} \leq 5) = \frac{6+8}{19} = \frac{14}{19}$.

❹ This measures the proportion of the area enclosed by the triangle, and the first and second rectangle, and so $\mathbf{P}(\mathbf{X} \leq 6) = \frac{6+8+2}{19} = \frac{16}{19}$.

❺ The area sought is that of a trapezium. One (of many possible ways to obtain this) is to observe that

$$\mathbf{P}(1 \leq \mathbf{X} \leq 2) = \mathbf{P}(\mathbf{X} \leq 2) - \mathbf{P}(\mathbf{X} \leq 1).$$

To find $\mathbf{P}(\mathbf{X} \leq 2)$ observe that the triangle with base on $[0;4]$ is similar to the one with base on $[0;2]$. If its height is h_1 then $\frac{h_1}{4} = \frac{2}{3}$, whence $h_1 = \frac{8}{3}$, and

$$\mathbf{P}(\mathbf{X} \leq 2) = \frac{1}{19}\left(\frac{1}{2} \cdot 2 \cdot \frac{8}{3}\right) = \frac{8}{57}.$$

To find $\mathbf{P}(\mathbf{X} \leq 1)$ observe that the triangle with base on $[0;4]$ is similar to the one with base on $[0;1]$. If its height is h_2 then $\frac{h_2}{4} = \frac{1}{3}$, whence $h_2 = \frac{4}{3}$, and

$$\mathbf{P}(\mathbf{X} \leq 1) = \frac{1}{19}\left(\frac{1}{2} \cdot 1 \cdot \frac{4}{3}\right) = \frac{2}{57}.$$

Finally,

$$\mathbf{P}(1 \leq \mathbf{X} \leq 2) = \mathbf{P}(\mathbf{X} \leq 2) - \mathbf{P}(\mathbf{X} \leq 1) = \frac{8}{57} - \frac{2}{57} = \frac{2}{19}.$$

❻ Since the curve does not extend from $x = 7$, we have

$$\mathbf{P}(\mathbf{X} \geq 6) = \mathbf{P}(6 \leq \mathbf{X} \leq 7) = \frac{2}{19}.$$

❼ From parts (2) and (3), $3 < m < 5$. For m in this range, a rectangle with base $m - 3$ and height 4 has area $4(m-3)$. Thus we need to solve

$$\frac{1}{2} = \mathbf{P}(\mathbf{X} \leq m) = \frac{6 + 4(m-3)}{19},$$

which implies

$$\frac{19}{2} = 6 + 4(m-3) \implies m = \frac{31}{8} = 3.875.$$

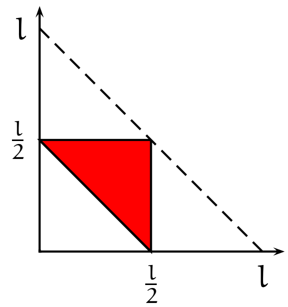


Figure 5.4: Example 527

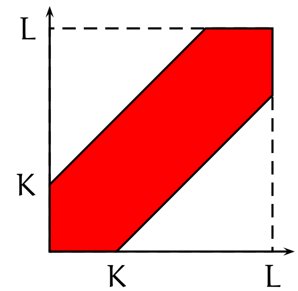


Figure 5.5: Example 528

527 Example A rod of length l is broken into three parts. What is the probability that these parts form a triangle?

Solution: Let x, y, and $l-x-y$ be the lengths of the three parts of the rod. If these parts are to form a triangle, then the triangle inequality must be satisfied, that is, the sum of any two sides of the triangle must be greater than the third. So we simultaneously must have

$$x+y>l-x-y \implies x+y>\frac{l}{2},$$

$$x+l-x-y>y \implies y<\frac{l}{2},$$

$$y+l-x-y>x \implies x<\frac{l}{2}.$$

Since trivially $0 \leq x+y \leq l$, what we are asking is for the ratio of the area of the region

$$\mathcal{A}=\{(x,y):0<x<\frac{l}{2},0<y<\frac{l}{2},x+y>\frac{l}{2}\}$$

to that of the triangle with vertices at $(0,0)$, $(l,0)$ and $(0,l)$. This is depicted in figure 5.4. The desired probability is thus

$$\frac{\frac{l^2}{8}}{\frac{l^2}{2}}=\frac{1}{4}.$$

528 Example Two points are chosen at random on a segment of length L. Find the probability that the distance between the points is at most K, where $0 < K < L$.

Solution: Let the points chosen be $\mathbf{X}$ and $\mathbf{Y}$ with $0 \leq \mathbf{X} \leq L$, $0 \leq \mathbf{Y} \leq L$, as in figure 5.5. The distance of the points is at most K if $|\mathbf{X} - \mathbf{Y}| \leq K$, that is

$$\mathbf{X} - K \leq \mathbf{Y} \leq \mathbf{X} + K.$$

The required probability is the ratio of the area shaded inside the square to the area of the square:

$$\frac{L^2 - 2\frac{(K-L)^2}{2}}{L^2} = \frac{K(2L-K)}{L^2}.$$

529 Example The amount 2.5 is split into two nonnegative real numbers uniformly at random, for instance, into 2.03 and 0.47 or into $2.5 - \sqrt{3}$ and $\sqrt{3}$. Then each of the parts is rounded to the nearest integer, for instance 2 and 0 in the first case above and 1 and 2 in the second. What is the probability that the two numbers so obtained will add up to 3?

Solution: Consider x and y with $0 \leq x \leq 2.5$ and $x + y = 2.5$ Observe that the sample space has size 2.5. We have a successful pair (x, y) if it happens that $(x, y) \in [0.5; 1] \times [1.5; 2]$ or $(x, y) \in [1.5; 2] \times [0.5; 1]$ The measure of all successful x is thus $0.5 + 0.5 = 1$. The probability sought is thus $\dfrac{1}{2.5} = \dfrac{2}{5}$.